Flow Typing for Lightweight Linearity (Appendix)

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A Translation Rules

In this section, we enumerate the translation rules from the flow typing system to the linear lambda calculus.

$$\frac{}{\Gamma, x : A \vdash x : A \dashv \Gamma}\ \textsc{Var} \quad \mapsto (x, \mathrm{pack}\,\Gamma)$$

$$\frac{\Gamma_1, x : A, y : B, \Gamma_2 \vdash e : C \dashv \Gamma_3}{\Gamma_1, y : B, x : A, \Gamma_2 \vdash e : C \dashv \Gamma_3}\ \textsc{Exch-L} \quad \mapsto \hat{e}$$

$$\frac{\Gamma_1 \vdash e : A \dashv \Gamma_2, x : B, y : C, \Gamma_3}{\Gamma_1 \vdash e : A \dashv \Gamma_2, y : C, x : B, \Gamma_3}\ \textsc{Exch-R}$$

$$\mapsto \mathrm{unpack}\ \hat{e}\ \mathrm{as}\ z, (\Gamma_2, x : B, y : C, \Gamma_3)\ \mathrm{in}\ (z, \mathrm{pack}\,(\Gamma_2, y : C, x : B, \Gamma_3))$$

$$\frac{\Gamma_1 \vdash e_1 : A \dashv \Gamma_2 \qquad \Gamma_2, x : A \vdash e_2 : B \dashv \Gamma_3}{\Gamma_1 \vdash (\mathrm{let}\ x = e_1\ \mathrm{in}\ e_2) : B \dashv \Gamma_3}\ \textsc{Let}$$

$$\mapsto \mathrm{unpack}\ \hat{e}_1\ \mathrm{as}\ x, \Gamma_2\ \mathrm{in}\ \hat{e}_2$$

$$\frac{\Gamma_1 \vdash e : A \dashv \Gamma_2}{\Gamma_1 \vdash \mathrm{true}(e) : \mathbf{t} \dashv \Gamma_2}\ \mathbf{t}\text{-I}$$

$$\mapsto \mathrm{unpack}\ \hat{e}\ \mathrm{as}\ x, \Gamma_2\ \mathrm{in}\ (\mathrm{true}(x), \mathrm{pack}\,\Gamma_2)$$

$$\frac{\Gamma_1 \vdash e : \mathbf{f} \dashv \Gamma_2}{\Gamma_1 \vdash \mathrm{false}_B(e) : B \dashv \Gamma_2}\ \mathbf{f}\text{-E}$$

$$\mapsto \mathrm{unpack}\ \hat{e}\ \mathrm{as}\ x, \Gamma_2\ \mathrm{in}\ (\mathrm{false}_B(x), \mathrm{pack}\,\Gamma_2)$$

$$\frac{\Gamma_2, x : A \vdash e : B \dashv \diamond}{\Gamma_1, \Gamma_2 \vdash (\lambda x : A.\, e) : A \multimap B \dashv \Gamma_1}\ \multimap\text{-I}$$

$$\mapsto ((\lambda x : A.\ \mathrm{unpack}\ \hat{e}\ \mathrm{as}\ y, \diamond\ \mathrm{in}\ y), \mathrm{pack}\,\Gamma_1)$$

$$\frac{\Gamma_1 \vdash e_1 : A \multimap B \dashv \Gamma_2 \qquad \Gamma_2 \vdash e_2 : A \dashv \Gamma_3}{\Gamma_1 \vdash e_1\ e_2 : B \dashv \Gamma_3}\ \multimap\text{-E}$$

$$\mapsto \mathrm{unpack}\ \hat{e}_1\ \mathrm{as}\ x, \Gamma_2\ \mathrm{in}\ \mathrm{unpack}\ \hat{e}_2\ \mathrm{as}\ y, \Gamma_3\ \mathrm{in}\ (x\ y, \mathrm{pack}\,\Gamma_3)$$

$$\frac{}{\Gamma \vdash * : I \dashv \Gamma}\ I\text{-I} \quad \mapsto (*, \mathrm{pack}\,\Gamma)$$

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$$\frac{\Gamma_1 \vdash e_1 : I \dashv \Gamma_2 \qquad \Gamma_2 \vdash e_2 : A \dashv \Gamma_3}{\Gamma_1 \vdash (\mathrm{let} * = e_1 \mathrm{~in~} e_2) : A \dashv \Gamma_3}\ I\text{-E}$$
$$\mapsto \mathrm{unpack}\ \hat{e}_1 \mathrm{~as~} x, \Gamma_2 \mathrm{~in~let} * = x \mathrm{~in~} \hat{e}_2$$

$$\frac{\Gamma_1 \vdash e_1 : A \dashv \Gamma_2 \qquad \Gamma_2 \vdash e_2 : B \dashv \Gamma_3}{\Gamma_1 \vdash (e_1, e_2) : A \otimes B \dashv \Gamma_3}\ \otimes\text{-I}$$
$$\mapsto \mathrm{unpack}\ \hat{e}_1 \mathrm{~as~} x, \Gamma_2 \mathrm{~in~unpack~} \hat{e}_2 \mathrm{~as~} y, \Gamma_3 \mathrm{~in~} ((x, y), \mathrm{pack}\ \Gamma_3)$$

$$\frac{\Gamma_1 \vdash e_1 : A \otimes B \dashv \Gamma_2 \qquad \Gamma_2, x : A, y : B \vdash e_2 : C \dashv \Gamma_3}{\Gamma_1 \vdash (\mathrm{let}\ (x, y) = e_1 \mathrm{~in~} e_2) : C \dashv \Gamma_3}\ \otimes\text{-E}$$
$$\mapsto \mathrm{unpack}\ \hat{e}_1 \mathrm{~as~} z, \Gamma_2 \mathrm{~in~let}\ (x, y) = z \mathrm{~in~} \hat{e}_2$$

$$\frac{\Gamma_2 \vdash e_1 : A \dashv \diamond \qquad \Gamma_2 \vdash e_2 : B \dashv \diamond}{\Gamma_1, \Gamma_2 \vdash \langle e_1, e_2 \rangle : A \mathbin{\&} B \dashv \Gamma_1}\ \&\text{-I}$$
$$\mapsto (\langle (\mathrm{unpack}\ \hat{e}_1 \mathrm{~as~} x, \diamond \mathrm{~in~} x), (\mathrm{unpack}\ \hat{e}_2 \mathrm{~as~} y, \diamond \mathrm{~in~} y) \rangle, \mathrm{pack}\ \Gamma_1)$$

$$\frac{\Gamma_1 \vdash e : A \mathbin{\&} B \dashv \Gamma_2}{\Gamma_1 \vdash \mathrm{fst}(e) : A \dashv \Gamma_2}\ \&\text{-EL}$$
$$\mapsto \mathrm{unpack}\ \hat{e} \mathrm{~as~} x, \Gamma_2 \mathrm{~in~} (\mathrm{fst}(x), \mathrm{pack}\ \Gamma_2)$$

$$\frac{\Gamma_1 \vdash e : A \mathbin{\&} B \dashv \Gamma_2}{\Gamma_1 \vdash \mathrm{snd}(e) : B \dashv \Gamma_2}\ \&\text{-ER}$$
$$\mapsto \mathrm{unpack}\ \hat{e} \mathrm{~as~} x, \Gamma_2 \mathrm{~in~} (\mathrm{snd}(x), \mathrm{pack}\ \Gamma_2)$$

$$\frac{\Gamma_1 \vdash e : A \dashv \Gamma_2}{\Gamma_1 \vdash \mathrm{inl}_{A \oplus B}(e) : A \oplus B \dashv \Gamma_2}\ \oplus\text{-IL}$$
$$\mapsto \mathrm{unpack}\ \hat{e} \mathrm{~as~} x, \Gamma_2 \mathrm{~in~} (\mathrm{inl}_{A \oplus B}(x), \mathrm{pack}\ \Gamma_2)$$

$$\frac{\Gamma_1 \vdash e : B \dashv \Gamma_2}{\Gamma_1 \vdash \mathrm{inr}_{A \oplus B}(e) : A \oplus B \dashv \Gamma_2}\ \oplus\text{-IR}$$
$$\mapsto \mathrm{unpack}\ \hat{e} \mathrm{~as~} x, \Gamma_2 \mathrm{~in~} (\mathrm{inr}_{A \oplus B}(x), \mathrm{pack}\ \Gamma_2)$$

$$\frac{\Gamma_1 \vdash e_1 : A \oplus B \dashv \Gamma_2 \qquad \Gamma_2, x : A \vdash e_2 : C \dashv \Gamma_3 \qquad \Gamma_2, y : B \vdash e_3 : C \dashv \Gamma_3}{\Gamma_1 \vdash (\mathrm{case}\ e_1 \mathrm{~of~inl}(x) \rightarrow e_2 \mid \mathrm{inr}(y) \rightarrow e_3) : C \dashv \Gamma_3}\ \oplus\text{-E}$$
$$\mapsto \mathrm{unpack}\ \hat{e}_1 \mathrm{~as~} \Gamma_2, z \mathrm{~in~case~} z \mathrm{~of~inl}(x) \rightarrow \hat{e}_2 \mid \mathrm{inr}(y) \rightarrow \hat{e}_3$$

$$\frac{\Gamma_1 \vdash e_1 : !A_1 \dashv \Gamma_2 \qquad \cdots \qquad \Gamma_n \vdash e_n : !A_n \dashv \Gamma_{n+1} \qquad x_1 : !A_1, \ldots, x_n : !A_n \vdash e : B \dashv \diamond}{\Gamma_1 \vdash \mathrm{promote}\ e_1, \ldots, e_n \mathrm{~for~} x_1, \ldots, x_n \mathrm{~in~} e : !B \dashv \Gamma_{n+1}}\ \textsc{Prom}$$
$$\mapsto \mathrm{unpack}\ \hat{e}_1 \mathrm{~as~} y_1, \Gamma_2 \mathrm{~in} \ldots \mathrm{unpack}\ \hat{e}_n \mathrm{~as~} y_n, \Gamma_{n+1} \mathrm{~in}$$
$$((\mathrm{promote}\ y_1, \ldots, y_n \mathrm{~for~} x_1, \ldots, x_n \mathrm{~in~} (\mathrm{unpack}\ \hat{e} \mathrm{~as~} z, \diamond \mathrm{~in~} z)), \mathrm{pack}\ \Gamma_{n+1})$$

$$\frac{\Gamma_1 \vdash e_1 : !A \dashv \Gamma_2 \qquad \Gamma_2 \vdash e_2 : B \dashv \Gamma_3}{\Gamma_1 \vdash \mathrm{discard}\, e_1 \,\mathrm{in}\, e_2 : B \dashv \Gamma_3}\ \textsc{Disc}$$

$$\mapsto \mathrm{unpack}\, \hat{e}_1 \,\mathrm{as}\, x, \Gamma_2 \,\mathrm{in}\, \mathrm{discard}\, x \,\mathrm{in}\, \hat{e}_2$$

$$\frac{\Gamma_1 \vdash e_1 : !A \dashv \Gamma_2 \qquad \Gamma_2, x : !A, y : !A \vdash e_2 : B \dashv \Gamma_3}{\Gamma_1 \vdash \mathrm{copy}\, e_1 \,\mathrm{as}\, x, y \,\mathrm{in}\, e_2 : B \dashv \Gamma_3}\ \textsc{Copy}$$

$$\mapsto \mathrm{unpack}\, \hat{e}_1 \,\mathrm{as}\, z, \Gamma_2 \,\mathrm{in}\, \mathrm{copy}\, z \,\mathrm{as}\, x, y \,\mathrm{in}\, \hat{e}_2$$

$$\frac{\Gamma_1 \vdash e : !A \dashv \Gamma_2}{\Gamma_1 \vdash \mathrm{derelict}\, e : A \dashv \Gamma_2}\ \textsc{Der}$$

$$\mapsto \mathrm{unpack}\, \hat{e} \,\mathrm{as}\, x, \Gamma_2 \,\mathrm{in}\, (\mathrm{derelict}\, x, \mathrm{pack}\, \Gamma_2)$$

$$\frac{\Gamma_1, x : A, y : B \vdash e : C \dashv \Gamma_2, x : A, y : B}{\Gamma_1, z : A \otimes B \vdash (\mathrm{let}\, \&(x, y) = \&z \,\mathrm{in}\, e) : C \dashv \Gamma_2, z : A \otimes B}\ \&\otimes\text{-E}$$

$$\mapsto \mathrm{let}\, (x, y) = z \,\mathrm{in}\, \mathrm{unpack}\, \hat{e} \,\mathrm{as}\, t, (\Gamma_2, x : A, y : B) \,\mathrm{in}\, \mathrm{let}\, z = (x, y) \,\mathrm{in}\, (t, \mathrm{pack}\, (\Gamma_2, z : A \otimes B))$$

$$\frac{\Gamma_1, x : A \vdash e_1 : C \dashv \Gamma_2, x : A \qquad \Gamma_1, y : B \vdash e_2 : C \dashv \Gamma_2, y : B}{\Gamma_1, z : A \oplus B \vdash (\mathrm{case}\, \&z \,\mathrm{of}\, \&\mathrm{inl}(x) \to e_1 \mid \&\mathrm{inr}(y) \to e_2) : C \dashv \Gamma_2, z : A \oplus B}\ \&\oplus\text{-E}$$

$$\begin{aligned}\mapsto\ & \mathrm{case}\, z \,\mathrm{of} \\ & \mid \mathrm{inl}(x) \to \mathrm{unpack}\, \hat{e}_1 \,\mathrm{as}\, t, (\Gamma_2, x : A) \,\mathrm{in}\, \mathrm{let}\, z = \mathrm{inl}(x) \,\mathrm{in}\, (t, \mathrm{pack}\, (\Gamma_2, z : A \oplus B)) \\ & \mid \mathrm{inr}(y) \to \mathrm{unpack}\, \hat{e}_2 \,\mathrm{as}\, t, (\Gamma_2, y : B) \,\mathrm{in}\, \mathrm{let}\, z = \mathrm{inr}(y) \,\mathrm{in}\, (t, \mathrm{pack}\, (\Gamma_2, z : A \oplus B))\end{aligned}$$

$$\frac{}{\Gamma_1, x : \mathbf{f} \vdash \mathrm{false}_B(\&x) : B \dashv \Gamma_1, x : \mathbf{f}}\ \&\mathbf{f}\text{-E}$$

$$\mapsto \mathrm{let}\, (t, x) = \mathrm{false}_{(B \otimes \mathbf{f})}(x) \,\mathrm{in}\, (t, \mathrm{pack}\, (\Gamma_1, x : \mathbf{f}))$$

$$\frac{\Gamma_1 \vdash e : A \dashv \Gamma_2}{\Gamma_1, x : I \vdash (\mathrm{let}\, \&{*} = \&x \,\mathrm{in}\, e) : A \dashv \Gamma_2, x : I}\ \&I\text{-E}$$

$$\mapsto \mathrm{unpack}\, \hat{e} \,\mathrm{as}\, t, \Gamma_2 \,\mathrm{in}\, (t, \mathrm{pack}\, (\Gamma_2, x : I))$$

$$\frac{\Gamma_2, x : A \vdash e : B \dashv x : A}{\Gamma_1, \Gamma_2 \vdash (\lambda \&x : A.\, e) : A \overset{\&}{\multimap} B \dashv \Gamma_1}\ \overset{\&}{\multimap}\text{-I}$$

$$\mapsto ((\lambda x : A.\, (\mathrm{unpack}\, \hat{e} \,\mathrm{as}\, y, (x : A) \,\mathrm{in}\, (y, x))), \mathrm{pack}\, \Gamma_1)$$

$$\frac{\Gamma_1 \vdash e : A \overset{\&}{\multimap} B \dashv \Gamma_2, x : A}{\Gamma_1 \vdash e\, \&x : B \dashv \Gamma_2, x : A}\ \overset{\&}{\multimap}\text{-E}$$

$$\mapsto \mathrm{unpack}\, \hat{e} \,\mathrm{as}\, y, (\Gamma_2, x : A) \,\mathrm{in}\, \mathrm{let}\, (z, x) = y\, x \,\mathrm{in}\, (z, \mathrm{pack}\, (\Gamma_2, x : A))$$

B String Diagram Semantics

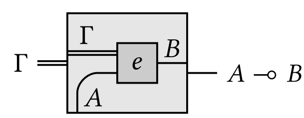
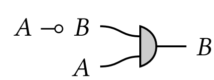
In this section, we describe the semantics of all of the linear lambda calculus rules and the flow typing rules in terms of string diagrams. A comprehensive introduction to string diagrams can be found in [2], and an introduction to their use specifically for functional computation can be found in [1]. The latter article, as well as [4], has inspired much of our notation.

Let $(\mathscr{C}, \otimes, I, \multimap)$ be a symmetric monoidal closed category (e.g. [3]). We assume without loss of generality that $\mathscr{C}$ is *strict* monoidal so that we can avoid explicit use of associators and unitors. One way to do this is described in [4]. We also assume any number of the following:

- binary coproducts $A + B$ with coprojections ι_1, ι_2;
- an initial object 0 with morphisms $?_A : 0 \to A$;
- binary products $A \times B$ with projections π_1, π_2;
- a terminal object $\top$ with morphisms $!_A : A \to \top$;
- a comonad $\mathbb{G} = (G, \epsilon, \delta)$ where $\epsilon : G \to 1_{\mathscr{C}}$ is the counit and $\delta : G \to G^2$ is the comultiplication, together with the following designated morphisms (subject to no laws):

$$\mathrm{wkg} : GA \to I; \quad \mathrm{ctr} : GA \to GA \otimes GA; \quad \mathrm{join}_0 : I \to GI; \quad \mathrm{join}_2 : GA \otimes GB \to G(A \otimes B)$$

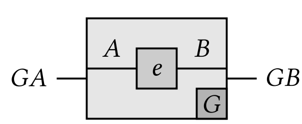
For abstraction and application we introduce the following additional string diagram notation:

abstraction	$\widetilde{e} : \Gamma \to (A \multimap B)$ where $e : \Gamma \otimes A \to B$	$\mapsto$	
application	$\varepsilon : (A \multimap B) \otimes A \to B$, the counit of $(-) \otimes A \dashv A \multimap (-)$	$\mapsto$	

For the comonad $\mathbb{G}$, we extend join_0 and join_2 to arbitrary n by iteration, assuming that (for concreteness) the applications of join are associated to the left:

$$\mathrm{join}_n : GA_1 \otimes \cdots \otimes GA_n \to G(A_1 \otimes \cdots \otimes A_n)$$

We additionally use boxes to signify application of functors. If $e : A \to B$, we write $Ge : GA \to GB$ using the following notation:

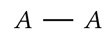


We translate types into the string diagram semantics as follows:

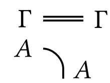
- pairs $A \otimes B$ are translated using the tensor product $A \otimes B$;
- the unit type I is translated as the monoidal unit I;
- sum types $A \oplus B$ are translated as the coproduct $A + B$;
- the empty type $\mathbf{f}$ is translated as the initial object 0;
- additive product types $A \mathbin{\&} B$ are translated as the product $A \times B$;
- the additive product unit $\mathbf{t}$ is translated as the terminal object $\top$;
- the exponential type $!A$ is translated as GA.

identity/variable

$$\frac{}{x : A \vdash x : A}$$

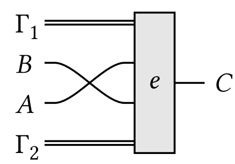


$$\frac{}{\Gamma, x : A \vdash x : A \dashv \Gamma}$$

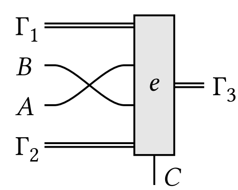


exchange

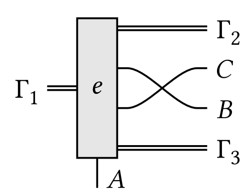
$$\frac{\Gamma_1, x : A, y : B, \Gamma_2 \vdash e : C}{\Gamma_1, y : B, x : A, \Gamma_2 \vdash e : C}$$



$$\frac{\Gamma_1, x : A, y : B, \Gamma_2 \vdash e : C \dashv \Gamma_3}{\Gamma_1, y : B, x : A, \Gamma_2 \vdash e : C \dashv \Gamma_3}$$

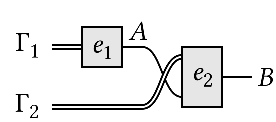


$$\frac{\Gamma_1 \vdash e : A \dashv \Gamma_2, x : B, y : C, \Gamma_3}{\Gamma_1 \vdash e : A \dashv \Gamma_2, y : C, x : B, \Gamma_3}$$

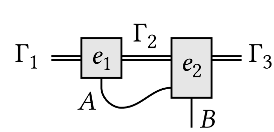


let

$$\frac{\Gamma_1 \vdash e_1 : A \qquad \Gamma_2, x : A \vdash e_2 : B}{\Gamma_1, \Gamma_2 \vdash (\mathrm{let}\, x = e_1 \,\mathrm{in}\, e_2) : B}$$

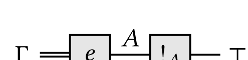


$$\frac{\Gamma_1 \vdash e_1 : A \dashv \Gamma_2 \qquad \Gamma_2, x : A \vdash e_2 : B \dashv \Gamma_3}{\Gamma_1 \vdash (\mathrm{let}\, x = e_1 \,\mathrm{in}\, e_2) : B \dashv \Gamma_3}$$

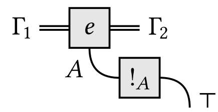


true

$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \mathrm{true}(e) : \mathbf{t}}$$

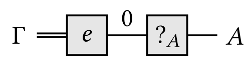


$$\frac{\Gamma_1 \vdash e : A \dashv \Gamma_2}{\Gamma_1 \vdash \mathrm{true}(e) : \mathbf{t} \dashv \Gamma_2}$$

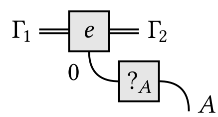


false

$$\frac{\Gamma \vdash e : \mathbf{f}}{\Gamma \vdash \mathrm{false}_A(e) : A}$$

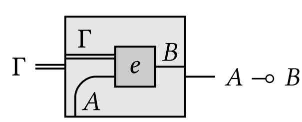


$$\frac{\Gamma_1 \vdash e : \mathbf{f} \dashv \Gamma_2}{\Gamma_1 \vdash \mathrm{false}_A(e) : A \dashv \Gamma_2}$$

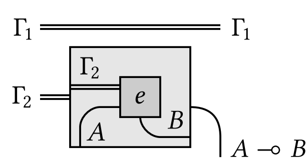


implication

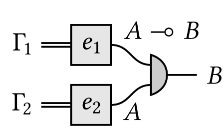
$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash (\lambda x : A.\, e) : A \multimap B}$$



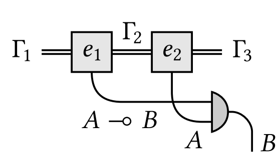
$$\frac{\Gamma_2, x : A \vdash e : B \dashv}{\Gamma_1, \Gamma_2 \vdash (\lambda x : A.\, e) : A \multimap B \dashv \Gamma_1}$$



$$\frac{\Gamma_1 \vdash e_1 : A \multimap B \qquad \Gamma_2 \vdash e_2 : A}{\Gamma_1, \Gamma_2 \vdash e_1\ e_2 : B}$$



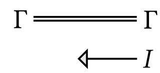
$$\frac{\Gamma_1 \vdash e_1 : A \multimap B \dashv \Gamma_2 \qquad \Gamma_2 \vdash e_2 : A \dashv \Gamma_3}{\Gamma_1 \vdash e_1\ e_2 : B \dashv \Gamma_3}$$



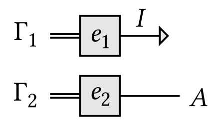
unit

$$\frac{}{\vdash * : I}$$

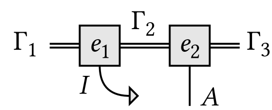
$$\frac{}{\Gamma \vdash * : I \dashv \Gamma}$$



$$\frac{\Gamma_1 \vdash e_1 : I \qquad \Gamma_2 \vdash e_2 : A}{\Gamma_1, \Gamma_2 \vdash (\text{let} * = e_1 \text{ in } e_2) : A}$$

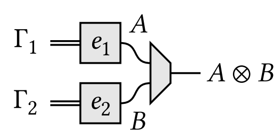


$$\frac{\Gamma_1 \vdash e_1 : I \dashv \Gamma_2 \qquad \Gamma_2 \vdash e_2 : A \dashv \Gamma_3}{\Gamma_1 \vdash (\text{let} * = e_1 \text{ in } e_2) : A \dashv \Gamma_3}$$

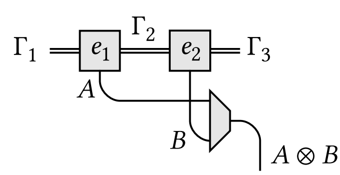


pair

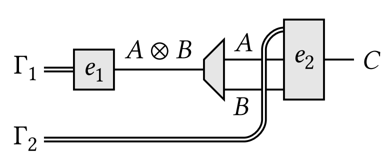
$$\frac{\Gamma_1 \vdash e_1 : A \qquad \Gamma_2 \vdash e_2 : B}{\Gamma_1, \Gamma_2 \vdash (e_1, e_2) : A \otimes B}$$



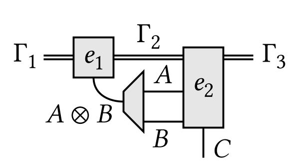
$$\frac{\Gamma_1 \vdash e_1 : A \dashv \Gamma_2 \qquad \Gamma_2 \vdash e_2 : B \dashv \Gamma_3}{\Gamma_1 \vdash (e_1, e_2) : A \otimes B \dashv \Gamma_3}$$



$$\frac{\Gamma_1 \vdash e_1 : A \otimes B \qquad \Gamma_2, x : A, y : B \vdash e_2 : C}{\Gamma_1, \Gamma_2 \vdash (\text{let } (x, y) = e_1 \text{ in } e_2) : C}$$



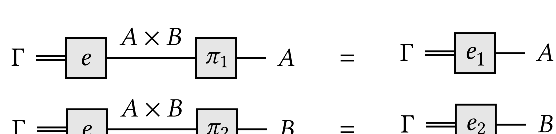
$$\frac{\begin{array}{c}\Gamma_1 \vdash e_1 : A \otimes B \dashv \Gamma_2 \\ \Gamma_2, x : A, y : B \vdash e_2 : C \dashv \Gamma_3\end{array}}{\Gamma_1 \vdash (\text{let } (x, y) = e_1 \text{ in } e_2) : C \dashv \Gamma_3}$$



additive pair

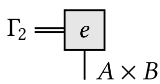
$$\frac{\Gamma \vdash e_1 : A \qquad \Gamma \vdash e_2 : B}{\Gamma \vdash \langle e_1, e_2 \rangle : A \,\&\, B}$$

e such that

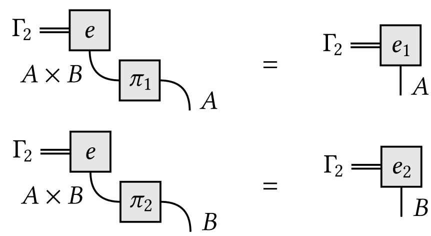


and

$$\frac{\Gamma_2 \vdash e_1 : A \dashv \qquad \Gamma_2 \vdash e_2 : B \dashv}{\Gamma_1, \Gamma_2 \vdash \langle e_1, e_2 \rangle : A \,\&\, B \dashv \Gamma_1}$$

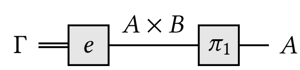


such that

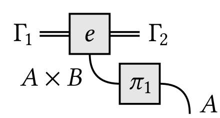


and

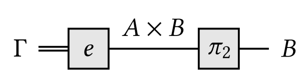
$$\frac{\Gamma \vdash e : A \,\&\, B}{\Gamma \vdash \mathrm{fst}(e) : A}$$



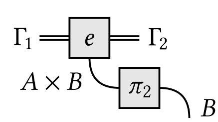
$$\frac{\Gamma_1 \vdash e : A \,\&\, B \dashv \Gamma_2}{\Gamma_1 \vdash \mathrm{fst}(e) : A \dashv \Gamma_2}$$



$$\frac{\Gamma \vdash e : A \,\&\, B}{\Gamma \vdash \mathrm{snd}(e) : B}$$

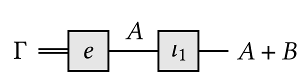


$$\frac{\Gamma_1 \vdash e : A \,\&\, B \dashv \Gamma_2}{\Gamma_1 \vdash \mathrm{snd}(e) : B \dashv \Gamma_2}$$

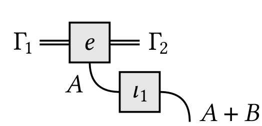


sum

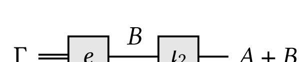
$$\frac{\Gamma \vdash e : A}{\Gamma \vdash \mathrm{inl}_{A \oplus B}(e) : A \oplus B}$$



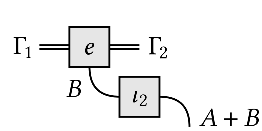
$$\frac{\Gamma_1 \vdash e : A \dashv \Gamma_2}{\Gamma_1 \vdash \mathrm{inl}_{A \oplus B}(e) : A \oplus B \dashv \Gamma_2}$$



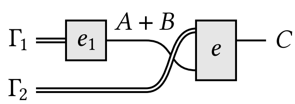
$$\frac{\Gamma \vdash e : B}{\Gamma \vdash \mathrm{inr}_{A \oplus B}(e) : A \oplus B}$$



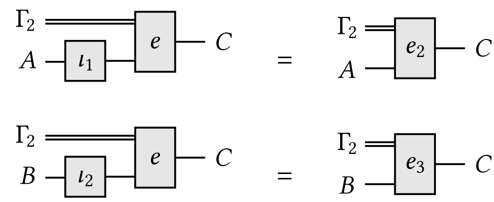
$$\frac{\Gamma_1 \vdash e : B \dashv \Gamma_2}{\Gamma_1 \vdash \mathrm{inr}_{A \oplus B}(e) : A \oplus B \dashv \Gamma_2}$$



$$\frac{\Gamma_1 \vdash e_1 : A \oplus B \qquad \Gamma_2, x : A \vdash e_2 : C \qquad \Gamma_2, y : B \vdash e_3 : C}{\Gamma_1, \Gamma_2 \vdash (\mathrm{case}\ e_1\ \mathrm{of}\ \mathrm{inl}(x) \longrightarrow e_2 \mid \mathrm{inr}(y) \longrightarrow e_3) : C}$$

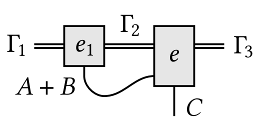


where

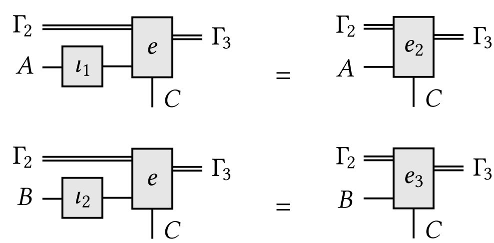


and

$$\frac{\Gamma_1 \vdash e_1 : A \oplus B \dashv \Gamma_2 \qquad \Gamma_2, x : A \vdash e_2 : C \dashv \Gamma_3 \qquad \Gamma_2, y : B \vdash e_3 : C \dashv \Gamma_3}{\Gamma_1 \vdash (\mathrm{case}\ e_1\ \mathrm{of}\ \mathrm{inl}(x) \longrightarrow e_2 \mid \mathrm{inr}(y) \longrightarrow e_3) : C \dashv \Gamma_3}$$



where

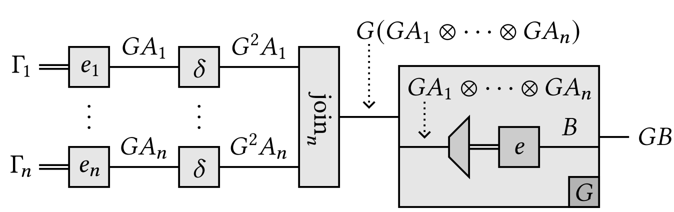


and

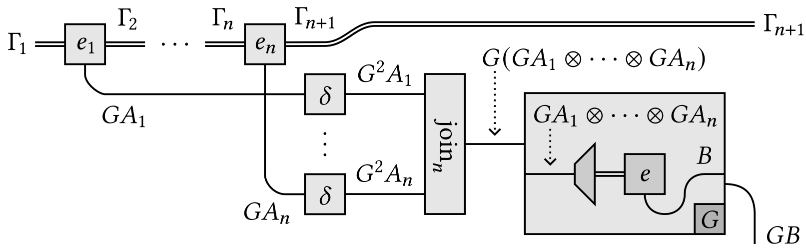
These definitions by universal property make use of the fact that $\Gamma_2 \otimes (A + B)$ is the coproduct of $\Gamma_2 \otimes A$ and $\Gamma_2 \otimes B$. This is true because $\otimes$ is a left adjoint and therefore commutes with coproducts.

exponential types

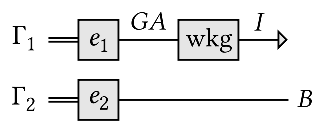
$$\frac{\Gamma_1 \vdash e_1 : !A_1 \qquad \cdots \qquad \Gamma_n \vdash e_n : !A_n \qquad x_1 : !A_1, , x_n : !A_n \vdash e : B}{\Gamma_1, \ldots, \Gamma_n \vdash \mathrm{promote}\ e_1, \ldots, e_n\ \mathrm{for}\ x_1, \ldots, x_n\ \mathrm{in}\ e : !B}$$



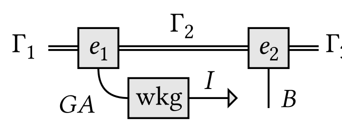
$$\frac{\Gamma_1 \vdash e_1 : !A_1 \dashv \Gamma_2 \qquad \cdots \qquad \Gamma_n \vdash e_n : !A_n \dashv \Gamma_{n+1} \qquad x_1 : !A_1, \ldots, x_n : !A_n \vdash e : B \dashv}{\Gamma_1 \vdash \mathrm{promote}\ e_1, \ldots, e_n\ \mathrm{for}\ x_1, \ldots, x_n\ \mathrm{in}\ e : !B \dashv \Gamma_{n+1}}$$



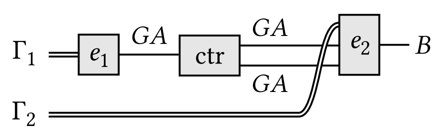
$$\frac{\Gamma_1 \vdash e_1 : !A \qquad \Gamma_2 \vdash e_2 : B}{\Gamma_1, \Gamma_2 \vdash \text{discard}\, e_1 \,\text{in}\, e_2}$$



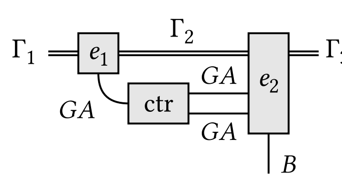
$$\frac{\Gamma_1 \vdash e_1 : !A \dashv \Gamma_2 \qquad \Gamma_2 \vdash e_2 : B \dashv \Gamma_3}{\Gamma_1 \vdash \text{discard}\, e_1 \,\text{in}\, e_2 : B \dashv \Gamma_3}$$



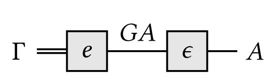
$$\frac{\Gamma_1 \vdash e_1 : !A \qquad \Gamma_2, x : !A, y : !A \vdash e_2 : B}{\Gamma_1, \Gamma_2 \vdash \text{copy}\, e_1 \,\text{as}\, x, y \,\text{in}\, e_2}$$



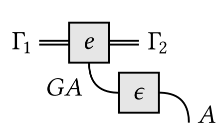
$$\frac{\Gamma_1 \vdash e_1 : !A \dashv \Gamma_2 \qquad \Gamma_2, x : !A, y : !A \vdash e_2 : B \dashv \Gamma_3}{\Gamma_1 \vdash \text{copy}\, e_1 \,\text{as}\, x, y \,\text{in}\, e_2 : B \dashv \Gamma_3}$$



$$\frac{\Gamma \vdash e : !A}{\Gamma \vdash \text{derelict}\, e : A}$$



$$\frac{\Gamma_1 \vdash e : !A \dashv \Gamma_2}{\Gamma_1 \vdash \text{derelict}\, e : A \dashv \Gamma_2}$$



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[3] Saunders Mac Lane. 1978. *Categories for the Working Mathematician*. Graduate Texts in Mathematics, Vol. 5. Springer New York, New York, NY. https://doi.org/10.1007/978-1-4757-4721-8

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