

Large Cardinals

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1 Inaccessible cardinals

1.1 Large cardinal properties

Modern set theory largely concerns itself with the consequences of the incompleteness phenomenon. Given any ‘reasonable’ set theory T , Gödel’s first incompleteness theorem shows that there is a sentence φ such that $T \not\vdash \varphi$ and $T \not\vdash \neg\varphi$. To be ‘reasonable’, the set of axioms must be computably enumerable, among other similar restrictions. In particular, Gödel’s second incompleteness theorem shows that $T \not\vdash \text{Con}(T)$, where $\text{Con}(T)$ is the statement that T is consistent. Hence,

$$\{\psi \mid T \vdash \psi\} \subsetneq \{\psi \mid T + \varphi \vdash \psi\}$$

We might say

$$T <_{\text{consequence}} T + \varphi$$

so T has strictly fewer consequences than $T + \varphi$. Modern set theory is about understanding the relation $\leq_{\text{consequence}}$ and other similar relations. It turns out that large cardinal axioms are the most natural hierarchy that we can use to measure the strength of set theories.

In this course we will not provide a definition for the notion of ‘large cardinal’, but we will provide an informal description. A *large cardinal property* is a formula Φ such that $\Phi(\kappa)$ implies that κ is a very large cardinal, so large that its existence cannot be proven in ZFC. A *large cardinal axiom* is an axiom of the form $\exists\kappa. \Phi(\kappa)$, which we will abbreviate ΦC . We begin with some non-examples.

- (i) κ is called an *aleph fixed point* if $\kappa = \aleph_\kappa$. Note that, for example, ω , ω_1 , and \aleph_ω are not aleph fixed points. However, we have the following result. We say that $F : \text{Ord} \rightarrow \text{Ord}$ is *normal* if $\alpha < \beta$ implies $F(\alpha) < F(\beta)$, and if λ is a limit, $F(\lambda) = \bigcup_{\alpha < \lambda} F(\alpha)$. One can show that every normal ordinal operation has arbitrarily large fixed points, and in particular that these fixed points may be enumerated by the ordinals. In particular, since the operation $\alpha \mapsto \aleph_\alpha$ is normal, it admits fixed points.
- (ii) Let $\Phi(\kappa)$ be the property

$$\kappa = \aleph_\kappa \wedge \text{Con}(\text{ZFC})$$

Clearly ΦC implies $\text{Con}(\text{ZFC})$, so $\text{ZFC} \not\vdash \Phi\text{C}$. We would like our large cardinal axioms to be unprovable by ZFC because of the size of the cardinal in question, not because of any other arbitrary reasons that we may attach to these axioms.

One source of large cardinal axioms is as follows. Consider the ordinal ω ; it is much larger than any ordinal smaller than it. We can consider properties that encapsulate the notion that ω is much larger than any smaller ordinal, and use these as large cardinal properties.

- (i) If $n < \omega$, then $n^+ < \omega$, where n^+ is the cardinal successor of n . We define

$$\Lambda(\kappa) \iff \forall\alpha. (\alpha < \kappa \rightarrow \alpha^+ < \kappa)$$

where α^+ is the least cardinal strictly larger than α . Then, $\Lambda(\kappa)$ holds precisely when κ is a limit cardinal. These need not be very large, for example, \aleph_ω is a limit cardinal, and the existence of this cardinal is proven by ZFC.

- (ii) If $n < \omega$, then $2^n < \omega$, where 2^n is the size of the power set of n .

$$\Sigma(\kappa) \iff \forall\alpha. (\alpha < \kappa \rightarrow 2^\alpha < \kappa)$$

where 2^α is the cardinality of $\mathcal{P}(\alpha)$. Such cardinals are called *strong limit cardinals*. We will show that these exist in all models of ZFC. Similarly to the aleph hierarchy, we can define the *beth* hierarchy, denoted \beth_α . This is given by

$$\beth_0 = \aleph_0; \quad \beth_{\alpha+1} = 2^{\beth_\alpha}; \quad \beth_\lambda = \bigcup_{\alpha < \lambda} \beth_\alpha$$

Cantor's theorem shows that $\aleph_\alpha \leq \beth_\alpha$, and the continuum hypothesis is the assertion that $\aleph_1 = \beth_1$. Note that κ is a strong limit cardinal if and only if $\kappa = \beth_\lambda$ for some limit ordinal λ . In particular, $\text{ZFC} \vdash \Sigma C$.

(iii) If $s : n \rightarrow \omega$ for $n < \omega$, then $\sup(s) = \bigcup \text{ran}(s) < \omega$. This gives rise to the following definition.

Definition. Let λ be a limit ordinal. We say that $C \subseteq \lambda$ is *cofinal* or *unbounded* if $\bigcup C = \lambda$. We define the *cofinality* of λ , denoted $\text{cf}(\lambda)$, to be the cardinality of the smallest cofinal subset. If λ is a cardinal, then $\text{cf}(\lambda) \leq \lambda$. If this inequality is strict, the cardinal is called *singular*; if this is an equality, it is called *regular*.

Note that if κ is regular, then if $\lambda < \kappa$, and for each $\alpha < \lambda$ we have a set $X_\alpha \subseteq \kappa$ of size $|X_\alpha| < \kappa$, then $\bigcup X_\alpha \neq \kappa$. It is easy to show that this property is equivalent to regularity.

We have therefore shown that ω is a regular cardinal. Note that \aleph_1 is also regular, since countable unions of countable sets are countable. This argument generalises to all successor cardinals, so all successor cardinals $\aleph_{\alpha+1}$ are regular. The cardinal \aleph_ω is not regular, as it is the union of $\{\aleph_n \mid n \in \mathbb{N}\}$, which is a subset of \aleph_ω of cardinality \aleph_0 , giving $\text{cf}(\aleph_\omega) = \aleph_0$. The cofinality of \aleph_{\aleph_ω} is also \aleph_0 . Limit cardinals are often singular.

1.2 Weakly inaccessible and inaccessible cardinals

Motivated by these examples of properties of ω , we make the following definition.

Definition. A cardinal κ is called *weakly inaccessible* if it is an uncountable regular limit, and (*strongly*) *inaccessible* if it is an uncountable regular strong limit. We write $\text{WI}(\kappa)$ to denote that κ is weakly inaccessible, and $\text{I}(\kappa)$ if κ is inaccessible.

To argue that these are large cardinal properties, we will show that they are very large, and that the existence of such cardinals cannot be proven in ZFC. Note that we cannot actually prove this statement; if ZFC were inconsistent, it would prove every statement. Whenever we write statements such as $\text{ZFC} \not\vdash \text{IC}$, it should be interpreted to mean ‘if ZFC is consistent, it does not prove IC’.

Many things in the relationship of WI and I are unclear: 2^{\aleph_0} is clearly not inaccessible as it is not a strong limit, but it is not clear that this is not a limit. The *generalised continuum hypothesis* GCH is that for all cardinals α , we have $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, and so $\aleph_\alpha = \beth_\alpha$. Under this assumption, the notions of limit and strong limit coincide, and so the notions of inaccessible cardinals and weakly inaccessible cardinals coincide.

Proposition. Weakly inaccessible cardinals are aleph fixed points.

Proof. Suppose κ is weakly inaccessible but $\kappa < \aleph_\kappa$. Fix α such that $\kappa = \aleph_\alpha$, then $\alpha < \kappa$. As κ is a limit cardinal, α must be a limit ordinal. But then $\aleph_\alpha = \bigcup_{\beta < \alpha} \aleph_\beta$, so in particular, the set $\{\aleph_\beta \mid \beta < \alpha\}$ is cofinal in κ , but this set is of size $|\alpha| < \kappa$. Hence κ is singular, contradicting regularity. \square

1.3 Second order replacement

We will now show that ZFC does not prove IC, and we omit the result for weakly inaccessible cardinals. We could do this via model-theoretic means: we assume $M \models \text{ZFC}$, and construct a model $N \models \text{ZFC} + \neg \text{IC}$. However, there is another approach we will take here. By Gödel's second incompleteness theorem, under the assumption that ZFC is consistent, we have $\text{ZFC} \not\vdash \text{Con}(\text{ZFC})$, so it suffices to show $\text{IC} \rightarrow \text{Con}(\text{ZFC})$. Gödel's completeness theorem states that $\text{Con}(T)$ holds if and only if there exists a model M with $M \models T$. Thus, it suffices to show that under the assumption that there is an inaccessible cardinal, we can construct a model of ZFC. Note that the metatheory in which the completeness theorem is proven actually matters; both theories and models are actually sets in the outer theory.

Recall that the *cumulative hierarchy* inside a model of set theory is given by

$$V_0 = \emptyset; \quad V_{\alpha+1} = \mathcal{P}(V_\alpha); \quad V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha$$

- (i) The axiom of foundation is equivalent to the statement that every set is an element of V_α for some α .
- (ii) (V_ω, \in) is a model of all of the axioms of set theory except for the axiom of infinity. This collection of axioms is called *finite set theory* FST.
- (iii) $(V_{\omega+\omega}, \in)$ is a model of all of the axioms of set theory except for the axiom of replacement. This theory is called *Zermelo set theory with choice* ZC. In fact, for any limit ordinal $\lambda > \omega$, ZFC proves that $(V_\lambda, \in) \models \text{ZC}$. That is, ZFC proves the existence of a model of ZC, or equivalently, $\text{ZFC} \vdash \text{Con}(\text{ZC})$. Hence, ZC cannot prove replacement, since Gödel's second incompleteness theorem applies to ZC. In this way, replacement behaves like a large cardinal axiom for ZC. The same holds for infinity and FST.

We briefly discuss why replacement fails in $V_{\omega+\omega}$. Consider the set of ordinals $\omega + n$ for $n < \omega$; this set does not belong to $V_{\omega+\omega}$ as its rank is $\omega + \omega$. However, the class function F given by $n \mapsto \omega + n$ is definable by a simple formula, and applying this to the set $\omega \in V_{\omega+\omega}$ gives a counterexample to replacement. Our counterexample is thus a cofinal subset of $V_{\omega+\omega}$ whose union does not lie in $V_{\omega+\omega}$. In some sense, the fact that $\omega + \omega$ is singular is the reason why $V_{\omega+\omega}$ does not satisfy replacement.

Now, consider $\alpha = \aleph_1$, which is regular. Consider $\mathcal{P}(\omega) \in V_{\omega+2} \subseteq V_{\omega_1}$. There is a definable surjection from $\mathcal{P}(\omega)$ to ω_1 , motivated by the proof of Hartogs' lemma. Indeed, subsets of ω can encode well-orders, and every countable well-order is encoded by a subset of ω , so the map

$$g : A \mapsto \begin{cases} \alpha & \text{if } A \text{ codes a well-order of order type } \alpha \\ 0 & \text{otherwise} \end{cases}$$

is a surjection $\mathcal{P}(\omega) \rightarrow \omega_1$. This class function has cofinal range in ω_1 , and so V_{ω_1} does not satisfy replacement.

We will prove that $\text{I}(\kappa)$ implies that V_κ models replacement. A set M is said to satisfy *second-order replacement* SOR if for every function $F : M \rightarrow M$ and every $x \in M$, the set $\{F(y) \mid y \in x\}$ is in M .

Any model of V_α that satisfies second-order replacement is a model of ZFC, as the counterexamples to replacement are special cases of violations of second-order replacement.

Theorem (Zermelo). If κ is inaccessible, then V_κ satisfies second-order replacement.

We first prove the following lemmas.

Lemma. If κ is inaccessible and $\lambda < \kappa$, then $|V_\lambda| < \kappa$.

Proof. This follows by induction. Note $|V_0| = 0 < \kappa$. If $|V_\alpha| < \kappa$, then as κ is a strong limit, $|V_{\alpha+1}| = |\mathcal{P}(V_\alpha)| = 2^{|V_\alpha|} < \kappa$. If λ is a limit and $|V_\alpha| < \kappa$ for all $\alpha < \lambda$, then if $|V_\lambda| = \kappa$, we have written κ as a union of less than κ sets of size less than κ , contradicting regularity. \square

Lemma. If κ is inaccessible and $x \in V_\kappa$, then $|x| < \kappa$.

Proof. Suppose $x \in V_\kappa = \bigcup_{\alpha < \kappa} V_\alpha$. Then there exists $\alpha < \kappa$ such that $x \in V_\alpha$. Then $x \subseteq V_\alpha$ as the V_α are transitive, but then $|x| \leq |V_\alpha| < \kappa$. \square

We can now prove Zermelo's theorem.

Proof. Let $F : V_\kappa \rightarrow V_\kappa$, and $x \in V_\kappa$; we must show that $R = \{F(y) \mid y \in x\} \in V_\kappa$. By the second lemma above, $|x| < \kappa$, hence $|R| < \kappa$. For each $y \in x$, define α_y to be the rank of $F(y)$. This is an ordinal less than κ . Consider $A = \{\alpha_y \mid y \in x\}$; its cardinality is bounded by that of x , so $|A| < \kappa$. But as κ is regular, A is not cofinal, so there is $\gamma < \kappa$ such that $A \subseteq V_\gamma$. By definition, $R \subseteq V_\gamma$, so $R \in V_{\gamma+1} \subseteq V_\kappa$, as required. \square

The definition of inaccessibility is precisely what is needed for this proof to work. The following converse holds.

Theorem (Shepherdson). If V_κ satisfies second-order replacement, then κ is inaccessible.

Proof. Suppose κ is not inaccessible, so either κ is singular or there is $\lambda < \kappa$ such that $2^\lambda \geq \kappa$. If κ is singular, then $\kappa = \bigcup_{\alpha < \lambda} \kappa_\alpha$ for $\lambda < \kappa$ and $\kappa_\alpha < \kappa$. Consider $C = \{\kappa_\alpha \mid \alpha < \lambda\}$; this set is cofinal in κ , but the cardinality of C is λ . Therefore, $C \notin V_\kappa$. We simply take the function $F : \alpha \mapsto \kappa_\alpha$, then the image of λ under F is $C \notin V_\kappa$, so F witnesses that V_κ violates second-order replacement.

Suppose there is $\lambda < \kappa$ such that $2^\lambda \geq \kappa$. Let $F : \mathcal{P}(\lambda) \rightarrow \kappa$ be a surjection. Since $\lambda < \kappa$, we must have $\mathcal{P}(\lambda) \in V_{\lambda+2} \subseteq V_\kappa$. Then the image of $\mathcal{P}(\lambda)$ under F is $\kappa \notin V_\kappa$ as required. \square

1.4 Countable transitive models of set theory

It is not generally the case that if $V_\kappa \models \text{ZFC}$ then κ is inaccessible. Moreover, the existence of an inaccessible cardinal is strictly stronger than the consistency of ZFC. We will show this second statement first.

Suppose κ is inaccessible, so $V_\kappa \models \text{ZFC}$. A standard model-theoretic argument shows there is a countable elementary substructure $(N, \in) \leq (V_\kappa, \in)$. In particular, $(N, \in) \models \text{ZFC}$. The proof of the downwards Löwenheim–Skolem theorem that we will use is a Skolem hull construction, given by

$$N_0 = \emptyset; \quad N_{k+1} = N_k \cup W(N_k); \quad N = \bigcup_{k \in \mathbb{N}} N_k$$

where $W(N_k)$ is a set of witnesses for all formulas of the form $\exists x. \varphi$ with parameters in N_k . The fact that this is an elementary substructure follows from the Tarski–Vaught test. We will now explore this model in more detail.

If $n \in \omega$, there is a formula φ_n such that $V_\kappa \models \varphi_n(x)$ if and only if $x = n$. Clearly, the formula $\exists x. \varphi_n(x)$ has precisely one witness, so $\omega \subseteq N_1$. Similarly, there are formulas $\varphi_\omega, \varphi_{\omega+\omega}, \varphi_{\omega \cdot 3}$ and so on. There is also a formula φ_{ω_1} such that $x = \omega_1$ if and only if $V_\kappa \models \varphi_{\omega_1}(x)$. As before, because there is a unique witness to this formula in V_κ , we must have $\omega_1 \in N_1$. But since the model N is countable, there must be a gap in the ordinals at some point below ω_1 . By the same argument, the model contains ω_2, ω_3 and so on. Therefore, N is a nontransitive model.

As (N, \in) is well-founded and extensional, by Mostowski’s collapsing theorem there is a unique transitive M such that $(M, \in) \cong (N, \in)$. This fills all of the gaps in our model. As this is an isomorphism, we obtain $(M, \in) \leq (N, \in) \leq (V_\kappa, \in)$, so (M, \in) is a countable transitive model of ZFC. In particular, its height $\alpha = \text{Ord} \cap M$ is a countable ordinal. There is an elementary embedding of M into V_κ given by the inverse of the Mostowski collapse. In particular, some $\beta < \alpha$ has the property that $M \models \varphi_{\omega_1}(\beta)$.

Therefore, the property ‘ x is a cardinal’ cannot be an *absolute* property between M and V_κ . A property is said to be absolute between M and some larger structure N if it holds in M precisely if it holds in N , where parameters are allowed to take values in the smaller structure M . If the truth of the property in the smaller structure implies the truth in the larger structure, we say the property is *upwards absolute*; conversely, if truth in the larger structure implies truth in the smaller one, we say the property is *downwards absolute*. The theory of absoluteness concerns the following classes of formulas, among others.

- (i) Δ_0 formulas, in which only bounded quantifiers are permitted, for example in ZFC, ‘ x is an ordinal’, ‘ f is a function’, ‘ x is a subset of y ’, ‘ x is ω ’.
- (ii) Σ_1 formulas, which are Δ_0 formulas surrounded by a single existential quantifier.
- (iii) Π_1 formulas, which are Δ_0 formulas surrounded by a single universal quantifier, for example ‘ x is a cardinal’ or ‘ x is the power set of y ’.

One can show that Δ_0 formulas are absolute between transitive models. Further, Σ_1 formulas are upwards absolute and Π_1 formulas are downwards absolute. The example above shows that ‘ x is a cardinal’ cannot be Δ_0 as it is not upwards absolute. Similarly, ‘ x is the power set of y ’ cannot be Δ_0 , because the object p that M believes is the power set of ω must be countable, and so cannot be the real power set in V_κ . As being a subset is absolute, this object p must consist of subsets of ω , but must only contain very few of them.

As being ω is Δ_0 , in fact all arithmetical statements (and therefore, by encoding, all syntactic statements) are Δ_0 .

Theorem. $\text{IC} \rightarrow \text{Con}(\text{ZFC})$ but $\text{Con}(\text{ZFC}) \not\rightarrow \text{IC}$.

Proof. The forward direction has already been proven. Since IC proves the consistency of ZFC, there is a countable transitive model $M \subseteq V_\kappa \subseteq V$ of ZFC. By absoluteness, $M \models \text{Con}(\text{ZFC})$, so $M \models \text{ZFC}^*$ where we define $\text{ZFC}^* = \text{ZFC} + \text{Con}(\text{ZFC})$. We have thus proven that IC implies the consistency of ZFC^* . So, by the second incompleteness theorem, $\text{ZFC}^* \not\vdash \text{IC}$. \square

1.5 Worldly cardinals

We now show that if $V_\kappa \models \text{ZFC}$, it is not necessarily the case that κ is inaccessible.

Observe that $M \neq V_\alpha$ for any α . Clearly $M \neq V_\omega$. But $|V_{\omega+1}| = |\mathcal{P}(\omega)| = 2^{\aleph_0}$, and $|V_\alpha| > 2^{\aleph_0}$ for all $\alpha > \omega + 1$. But M is countable, so it cannot be any of these.

Recall the definition of N by

$$N_0 = \emptyset; \quad N_{k+1} = W(N_k); \quad N = \bigcup_{k \in \mathbb{N}} N_k$$

We wish to create a similar structure that is of the form V_α for some α . We define

$$\alpha_0 = 0; \quad \alpha_{k+1} = \sup\{\text{rank}(x) \mid x \in W(V_{\alpha_k})\}; \quad \alpha = \sup\{\alpha_n \mid n \in \mathbb{N}\}$$

Note that $N \subseteq V_{\alpha_1}$.

Theorem. $V_\alpha \leq V_\kappa$ and $\alpha < \kappa$.

Proof. The first statement follows from the Tarski–Vaught test. To show $\alpha < \kappa$, we first show by induction that $\alpha_k < \kappa$. This is clearly true for $k = 0$. Now, if $\alpha_k < \kappa$, we have $|V_{\alpha_k}| < \kappa$ by a previous lemma. Thus,

$$|W(V_{\alpha_k})| \leq \aleph_0 \cdot |V_{\alpha_k}^{<\omega}| = |V_{\alpha_k}| < \kappa$$

where $X^{<\omega}$ is the set of finite sequences of elements of X . Hence $\{\text{rank}(x) \mid x \in W(V_{\alpha_k})\}$ is a set of less than κ ordinals less than κ , so it must be bounded by regularity. Finally, as α is a countable union of the α_k , regularity again shows $\alpha < \kappa$. \square

Remark. The ordinal α produced in this way has countable cofinality, so cannot be inaccessible. In particular, $V_\alpha \models \text{ZFC}$ but α is not inaccessible.

Definition. We call an ordinal α *worldly* if $V_\alpha \models \text{ZFC}$, and write $\text{Wor}(\alpha)$.

We have shown $\text{I}(\kappa) \rightarrow \text{Wor}(\kappa)$, but not the other way round given that a worldly cardinal exists. In particular,

$$\text{IC} \rightarrow \text{WorC} \rightarrow \text{Con}(\text{ZFC})$$

Theorem. If κ is a worldly ordinal, κ is a cardinal.

Proof. First, observe that κ is a limit ordinal; otherwise, its predecessor would be the largest ordinal in the model, but ZFC proves that there is no largest ordinal. Suppose κ is not a cardinal, so there is $\lambda < \kappa$ such that there is a bijection $\lambda \rightarrow \kappa$. In particular, $\lambda < \kappa < \lambda^+$. By the proof of Hartogs' lemma, there is a relation $R \subseteq \lambda \times \lambda$ such that $(\lambda, R) \cong (\kappa, \in)$. Assuming Kuratowski's definition

of ordered pairs, an element of $\lambda \times \lambda$ is an element of V_λ , so $\lambda \times \lambda \in V_{\lambda+1}$ and $R \in V_{\lambda+1}$. The pair (λ, R) is an element of $V_{\lambda+3} \subseteq V_\kappa$. Thus V_κ contains a well-order (λ, R) of order type κ . But ZFC proves that every well-ordering is isomorphic to a unique ordinal, so we must have $\kappa \in V_\kappa$, which is a contradiction. \square

1.6 The consistency strength hierarchy

Let B be a base theory; we will often use ZFC. If T, S are extensions of B , we say that T has lower *consistency strength* than S , written $T \leq_{\text{Con}} S$, if $B \vdash \text{Con}(S) \rightarrow \text{Con}(T)$. We say that T and S is *equiconsistent*, written $T \equiv_{\text{Con}} S$, if $T \leq_{\text{Con}} S$ and $S \leq_{\text{Con}} T$, and write $T <_{\text{Con}} S$ if $T \leq_{\text{Con}} S$ but $S \not\leq_{\text{Con}} T$.

Remark. (i) If I is inconsistent, then $T \leq_{\text{Con}} I$ for all T . All inconsistent theories are equiconsistent. In particular, T is consistent if and only if $T <_{\text{Con}} I$. We typically write \perp for an inconsistent theory.

(ii) $<_{\text{Con}}$ is more than just ‘proving more theorems’. If φ is such that $\text{ZFC} \not\vdash \varphi$ and $\text{ZFC} \not\vdash \neg\varphi$, it is not necessarily the case that $\text{ZFC} <_{\text{Con}} \text{ZFC} + \varphi$ or $\text{ZFC} <_{\text{Con}} \text{ZFC} + \neg\varphi$. For example, $\text{ZFC} + \text{CH}$, $\text{ZFC} + \neg\text{CH}$, and ZFC are all equiconsistent.

(iii) The second incompleteness theorem shows, for suitably nice theories T , that if $T \neq \perp$ then $T <_{\text{Con}} T + \text{Con}(T)$. Note that it is possible that T is consistent but $T + \text{Con}(T)$ is inconsistent, so the incompleteness theorem does not necessarily give an infinite chain of strict consistency strength inequalities. For example, consider

$$\text{ZFC}^\dagger = \text{ZFC} + \neg \text{Con}(\text{ZFC})$$

Since $\text{ZFC}^\dagger \supseteq \text{ZFC}$, we must have $\text{Con}(\text{ZFC}^\dagger) \rightarrow \text{Con}(\text{ZFC})$, but $\text{ZFC}^\dagger \rightarrow \neg \text{Con}(\text{ZFC})$, so $\text{ZFC}^\dagger + \text{Con}(\text{ZFC}^\dagger)$ is inconsistent.

In conclusion,

$$\text{ZFC} <_{\text{Con}} \text{ZFC} + \text{Con}(\text{ZFC}) <_{\text{Con}} \text{ZFC} + \text{WorC} <_{\text{Con}} \text{ZFC} + \text{IC}$$

where the second inequality uses the same argument as $\text{IC} \rightarrow \text{Con}(\text{ZFC} + \text{Con}(\text{ZFC}))$.

We will see that $\text{ZFC} \equiv_{\text{Con}} \text{ZFC} + \neg\text{IC}$. Many large cardinal axioms have this property that their negations are weak.

If κ is the least inaccessible cardinal, then V_κ is a model of ZFC, but we can show that it cannot satisfy IC. Note that the statement ‘ λ is inaccessible’ is a Π_1 statement, so is downwards absolute. Given a model with two inaccessible cardinals $\kappa_0 < \kappa_1$, we have $V_{\kappa_1} \models \text{ZFC} + \text{I}(\kappa_0)$ so in particular, $V_{\kappa_1} \models \text{ZFC} + \text{IC}$.

Lemma. If α is a limit ordinal, then the formula ‘ λ is inaccessible’ is absolute for V_α and V .

In particular, V_κ above does not satisfy IC.

Proof. By downwards absoluteness, it suffices to show that if $V_\alpha \models \text{I}(\lambda)$ then $\text{I}(\lambda)$. Suppose not, so λ is singular or not a strong limit.

Let λ be singular, so there is a cofinal set $C \subseteq \lambda$ with $|C| = \gamma < \lambda$, so there is a bijection $f : \gamma \rightarrow C$. Note that being singular is Σ_1 , witnessed by C, γ, f . We have $C \in V_{\lambda+1}$, $\gamma \in V_\lambda$, and $f \in V_{\lambda+2}$. All

of these are subsets of V_α , so these witnesses exist in V_α . Hence V_α believes that C is a cofinal set of cardinality less than λ , so it believes λ is singular, contradicting inaccessibility.

Now let λ not be a strong limit. Let $\gamma < \lambda$, and let $f : \mathcal{P}(\gamma) \rightarrow \lambda$ be a surjection. Then $\mathcal{P}(\gamma) \in V_{\gamma+2} \subseteq V_\lambda \subseteq V_\alpha$, and so this function is an element of $V_{\lambda+2} \subseteq V_\alpha$. The statement that it is a surjection is absolute, so V_α believes f is a surjection from $\mathcal{P}(\gamma)$ to λ , contradicting its belief that λ is a strong limit. \square

Therefore, we have the following.

Theorem. Suppose $ZFC + IC$, and let κ be the least inaccessible. Then $V_\kappa \models ZFC + \neg IC$.

Proof. Suppose $V_\kappa \models ZFC + IC$. Then there is $\lambda < \kappa$ such that $V_\kappa \models I(\lambda)$, but by the previous lemma this contradicts minimality of κ . \square

Therefore, we have the following.

$$ZFC + IC \vdash \text{there is a transitive model of } ZFC + \neg IC$$

For any theory T , we write

$$T^* = T + \text{Con}(T)$$

We make the following remarks.

- (i) Observe that if S proves that there is a transitive model of T , then $S \vdash \text{Con}(T^*)$ because consistency statements are downwards absolute between transitive models.
- (ii) Note also that if S proves every axiom of T , then $\text{Con}(S) \rightarrow \text{Con}(T)$.
- (iii) If T is not equiconsistent with \perp , then $\text{Con}(T) \not\leftrightarrow \text{Con}(T^*)$.

We can therefore show

$$\text{Con}(ZFC + \neg IC) \not\leftrightarrow \text{Con}(ZFC + IC)$$

assuming that $ZFC + \neg IC$ is consistent. We have that $ZFC + IC$ yields a transitive model of $ZFC + \neg IC$. Thus, by (i), $ZFC + IC$ implies $\text{Con}((ZFC + \neg IC)^*)$. Hence $\text{Con}(ZFC + IC) \rightarrow \text{Con}((ZFC + \neg IC)^*)$, so if the given implication were to hold, it would contradict Gödel's second incompleteness theorem. Thus, if $ZFC + \neg IC$ is consistent,

$$ZFC + \neg IC <_{\text{Con}} ZFC + IC$$

Observe that none of the proofs given in this section work for weakly inaccessible cardinals, so it is not clear that weakly inaccessible cardinals qualify as large cardinals. However, under the generalised continuum hypothesis, we have $\aleph_\alpha = \beth_\alpha$ and so the notions of weakly inaccessible cardinal and inaccessible cardinal coincide. In Part III Forcing and the Continuum Hypothesis, we see that if $M \models ZFC$, there is $L \subseteq M$ such that L is transitive in M , L contains all the ordinals of M , and $L \models ZFC + GCH$. Thus, given a model $M \models ZFC + WIC$, we obtain $L \models ZFC + IC$, and thus the two axioms WIC and IC are equiconsistent.

Note that 2^{\aleph_0} is not a strong limit, but it is consistent that 2^{\aleph_0} is weakly inaccessible (under suitable assumptions), so the notions of weakly inaccessible cardinals and inaccessible cardinals do not coincide.

2 Measurability and compactness

2.1 The measure problem

Let \mathbb{I} denote the unit interval $[0, 1] \subseteq \mathbb{R}$. A function $\mu : \mathcal{P}(\mathbb{I}) \rightarrow \mathbb{I}$ is called a *measure* if

- (i) $\mu(\mathbb{I}) = 1$ and $\mu(\emptyset) = 0$;
- (ii) (translation invariance) if $X \subseteq \mathbb{I}$, $r \in \mathbb{R}$, and $X + r = \{x + r \mid x \in X\} \subseteq \mathbb{I}$, then $\mu(X) = \mu(X + r)$; and
- (iii) (countable additivity) if $(A_n)_{n \in \mathbb{N}}$ is a family of pairwise disjoint subsets of \mathbb{I} , then $\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$.

The *Lebesgue measure problem* was the question of whether such a measure function exists. Vitali proved that a measure cannot be defined on all of $\mathcal{P}(\mathbb{I})$. This proof requires the axiom of choice nontrivially. In 1970, Solovay proved that if ZFC + IC is consistent, then, there is a model of ZF in which all sets are Lebesgue measurable. In 1984, Shelah showed that the inaccessible cardinal was necessary to construct this model.

Now, replace translation invariance with the requirement that for all $x \in \mathbb{I}$, we have $\mu(\{x\}) = 0$, and call such measures *Banach measures*. *Banach's measure problem* was the question of whether a Banach measure exists. Note that every Lebesgue measure is a Banach measure. If $\mu(\{x\}) > 0$ for some x , then by translation invariance, every singleton has the same measure $\mu(\{x\}) > 0$. There is some natural number n such that $n\mu(\{x\}) > 1$, but this contradicts countable additivity using a set of n reals. Observe that for any $\varepsilon > 0$, there can be only finitely many pairwise disjoint sets with measure at least ε .

Banach and Kuratowski proved in 1929 that the continuum hypothesis implies that there are no Banach measures on \mathbb{I} . We can define Banach measures on any set S by also replacing property (i) with the requirement that $\mu(S) = 1$ and $\mu(\emptyset) = 0$. Note that if $|S| = |S'|$, then there is a Banach measure on S if and only if there is one on S' . Thus, having a Banach measure is a property of cardinals.

For larger cardinals, it may not be natural to just consider countable additivity.

Definition. A Banach measure μ is called λ -*additive* if for all $\gamma < \lambda$ and pairwise disjoint families $\{A_\alpha \mid \alpha < \gamma\}$, then

$$\mu\left(\bigcup A_\alpha\right) = \sup \left\{ \sum_{\alpha \in F} \mu(A_\alpha) \mid F \subseteq \gamma \text{ finite} \right\}$$

Theorem. If κ is the smallest cardinal that has a Banach measure, then that measure is κ -additive.

2.2 Real-valued measurable cardinals

Definition. A cardinal κ is *real-valued measurable*, written $\text{RVM}(\kappa)$, if there is a κ -additive Banach measure on κ .

Proposition. Every real-valued measurable cardinal is regular.

Proof. Suppose that κ is a real-valued measurable cardinal, and that $C \subseteq \kappa$ is cofinal with $|C| = \lambda < \kappa$. We can write

$$C = \{\gamma_\alpha \mid \alpha < \lambda\}$$

where γ_α is increasing in α . Consider

$$C_\alpha = \{\xi \mid \gamma_\alpha \leq \xi < \gamma_{\alpha+1}\}$$

Then $\bigcup_{\alpha < \lambda} C_\alpha = \kappa$ as C is cofinal, and the C_α are disjoint. Note that $|C_\alpha| \leq |\gamma_{\alpha+1}| < \kappa$. Writing $C_\alpha = \bigcup_{x \in C_\alpha} \{x\}$, we observe by κ -additivity that $\mu(C_\alpha) = 0$. But again by κ -additivity, $\mu(\kappa) = 0$, contradicting property (i). \square

Proposition (the pigeonhole principle). Let κ be regular, $\lambda < \kappa$, and $f : \kappa \rightarrow \lambda$. Then there is some $\alpha \in \lambda$ such that $|f^{-1}(\alpha)| = \kappa$.

Proof. We have

$$\kappa = \bigcup_{\alpha \in \lambda} f^{-1}(\alpha)$$

giving the result immediately by regularity of κ . \square

Proposition. All successor cardinals are regular.

Proposition. If μ is a Banach measure on S , and C is a family of pairwise disjoint sets of positive μ -measure, then C is countable.

Proof. Consider the collection

$$C_n = \left\{ A \in C \mid \mu(A) > \frac{1}{n} \right\}$$

Observe that each C_n is finite, so $C = \bigcup_{n \in \mathbb{N}} C_n$ must be countable. \square

Lemma (Ulam). For any cardinal λ , there is an *Ulam matrix* A_α^ξ indexed by $\alpha < \lambda^+$ and $\xi < \lambda$ such that

- (i) for a given ξ , the set $\{A_\alpha^\xi \mid \alpha < \lambda^+\}$ is a pairwise disjoint family; and

(ii) for a given α , we have

$$\left| \lambda^+ \setminus \bigcup_{\xi < \lambda} A_\alpha^\xi \right| \leq \lambda$$

Proof. For each $\gamma < \lambda^+$, fix a surjection $f_\gamma : \lambda \rightarrow \gamma + 1$. Define

$$A_\alpha^\xi = \{\gamma \mid f_\gamma(\xi) = \alpha\}$$

It is clear that property (i) holds. For property (ii), suppose

$$\gamma \in \lambda^+ \setminus \bigcup_{\xi < \lambda} A_\alpha^\xi$$

Then $\gamma < \lambda^+$ and for all ξ , we have $f_\gamma(\xi) \neq \alpha$. Hence

$$\lambda^+ \setminus \bigcup_{\xi < \lambda} A_\alpha^\xi \subseteq \alpha$$

so the size of this set is at most λ . □

Theorem. Every real-valued measurable cardinal is weakly inaccessible.

Remark. If there is a Banach measure on $[0, 1]$, then in particular 2^{\aleph_0} is weakly inaccessible.

Proof. We have already shown regularity. Suppose κ is not a limit cardinal, so $\kappa = \lambda^+$. Let $(A_\alpha^\xi)_{\alpha < \lambda^+; \xi < \lambda}$ be an Ulam matrix for λ . By (ii),

$$|Z_\alpha| \leq \lambda; \quad Z_\alpha = \lambda^+ \setminus \bigcup_{\xi < \lambda} A_\alpha^\xi$$

so by κ -additivity, $\mu(Z) = 0$. Hence

$$\mu\left(\bigcup_{\xi < \lambda} A_\alpha^\xi\right) = 1$$

This is a small union of sets of measure 1, so again by κ -additivity there is some ξ_α such that $\mu(A_\alpha^{\xi_\alpha}) > 0$. Let $f : \lambda^+ \rightarrow \lambda$ be the map $\alpha \mapsto \xi_\alpha$. By the pigeonhole principle, there is some ξ and a set $A \subseteq \lambda^+$ with $|A| = \lambda^+$ such that for all $\alpha \in A$, we have $\xi_\alpha = \xi$. By property (i), the collection $\{A_\alpha^\xi \mid \alpha \in A\}$ is a collection of uncountable size λ^+ of pairwise disjoint sets, all of which have positive measure, but we have already shown that such a collection must be countable. □

2.3 Measurable cardinals

Definition. A Banach measure μ is called *two-valued* if μ takes values in $\{0, 1\}$.

This removes any mention of the real numbers from the definition of a Banach measure.

Remark. Two-valued measures correspond directly to ultrafilters. Recall that F is a *filter* on S if

- (i) $\emptyset \notin F, S \in F$;
- (ii) if $A \subseteq B$ then $A \in F \rightarrow B \in F$;
- (iii) if $A, B \in F$ then $A \cap B \in F$.

We say that F is an *ultrafilter* if $A \in F$ or $S \setminus A \in F$ for all $A \subseteq S$. F is *nonprincipal* if for all $x \in S$, the singleton $\{x\}$ is not in F . An ultrafilter is λ -complete if for all $\gamma < \lambda$ and all families $\{A_\alpha \mid \alpha < \gamma\} \subseteq F$, we have $\bigcap_{\alpha < \gamma} A_\alpha \in F$. In this way, the collection of sets of a two-valued Banach measure μ that are assigned measure 1 form a nonprincipal ultrafilter. This filter is λ -complete if and only if μ is λ -additive.

Definition. An uncountable cardinal κ is *measurable*, written $M(\kappa)$, if there is a κ -complete nonprincipal ultrafilter on κ .

- Remark.*
- (i) ZFC proves that there is an \aleph_0 -complete nonprincipal ultrafilter on \aleph_0 , because \aleph_0 -completeness is equivalent to closure under finite intersections, which is trivial.
 - (ii) A cardinal κ is called *Ulam measurable* if there is an \aleph_1 -complete nonprincipal ultrafilter on κ . With this definition, the least Ulam measurable cardinal is measurable. So the existence of an Ulam measurable cardinal is equivalent to the existence of a measurable cardinal.
 - (iii) The theories ZFC + MC and ZFC + RVMC are equiconsistent. This can be shown analogously to inaccessible and weakly inaccessible cardinals, this time using a variant of Gödel's constructible universe.

Theorem. Every measurable cardinal is inaccessible.

Proof. We have already shown regularity in the real-valued measurable cardinal case. Let κ be measurable with ultrafilter U . Suppose it is not a strong limit, so there is $\lambda < \kappa$ such that $2^\lambda \geq \kappa$. Then there is an injection $f : \kappa \rightarrow B_\lambda$, where B_λ is the set of functions $\lambda \rightarrow 2$. Fix some $\alpha < \lambda$, then for each $\gamma < \kappa$, either

$$f(\gamma)(\alpha) = 0 \text{ or } f(\gamma)(\alpha) = 1$$

Let

$$A_0^\alpha = \{\gamma \mid f(\gamma)(\alpha) = 0\}; \quad A_1^\alpha = \{\gamma \mid f(\gamma)(\alpha) = 1\}$$

These two sets are disjoint and have union κ . So there is exactly one number $b \in \{0, 1\}$ such that $A_b^\alpha \in U$. Define $c \in B_\lambda$ by $c(\alpha) = b$. Then

$$X_\alpha = A_{c(\alpha)}^\alpha \in U$$

This is a collection of λ -many sets that are all in U , so by κ -completeness, their intersection $\bigcap_{\alpha < \lambda} X_\alpha$ also lies in U . Suppose $\gamma \in \bigcap_{\alpha < \lambda} X_\alpha$, so for all $\alpha < \lambda$, we have $\gamma \in A_{c(\alpha)}^\alpha$. Equivalently, for all $\alpha < \lambda$, we have $f(\gamma)(\alpha) = c(\alpha)$. So γ lies in this intersection if and only if $f(\gamma)$ is precisely the function c . Hence

$$\bigcap_{\alpha < \lambda} X_\alpha \subseteq \{f^{-1}(c)\}$$

So this intersection has either zero or one element, and in particular, it is not in the ultrafilter, giving a contradiction. \square

Nonprincipal ultrafilters on κ are not κ^+ -complete, because κ itself is a union of κ -many singletons. Principal ultrafilters are complete for any cardinal. However, we can emulate completeness for non-principal ultrafilters at the cardinal κ^+ using the following method. If $(A_\alpha)_{\alpha \leq \kappa}$ is a sequence of subsets of κ , its *diagonal intersection* is

$$\Delta_{\alpha \leq \kappa} A_\alpha = \left\{ \xi \in \kappa \mid \xi \in \bigcap_{\alpha < \xi} A_\alpha \right\}$$

A filter on κ is called *normal* if it is closed under diagonal intersections.

Theorem. If κ is measurable, then there is a κ -complete normal nonprincipal ultrafilter on κ .

The proof will be given later, and is also on an example sheet.

2.4 Weakly compact cardinals

Let $[X]^n$ be the set of n -element subsets of X . A *2-colouring* of \mathbb{N} is a map $c : [\mathbb{N}]^2 \rightarrow \{\text{red}, \text{blue}\}$. Ramsey's theorem states that for each 2-colouring c , there is an infinite subset $X \subseteq \mathbb{N}$ such that $c|_{[X]^2}$ is *monochromatic* (or *homogeneous*): each 2-element subset is given the same colour under c .

This property is invariant under bijection, so this is really a property of the cardinal \aleph_0 . In *Erdős' arrow notation*, we write

$$\kappa \rightarrow (\lambda)_m^n$$

if for every colouring $c : [\kappa]^n \rightarrow m$, there is a monochromatic subset $X \subseteq \kappa$ of size λ :

$$|c[[X]^n]| = 1$$

In this notation, Ramsey's theorem becomes the statement

$$\aleph_0 \rightarrow (\aleph_0)_2^2$$

We can now make the following definition.

Definition. An uncountable cardinal κ is called *weakly compact*, written $W(\kappa)$, if $\kappa \rightarrow (\kappa)_2^2$.

The name will be explained later.

Theorem (Erdős). Every weakly compact cardinal is inaccessible.

Proof. Suppose κ is weakly compact but not regular. Then $\kappa = \bigcup_{\alpha < \lambda} X_\alpha$ for $\alpha < \kappa$ and disjoint sets X_α with $|X_\alpha| < \kappa$. We define a colouring c as follows. A pair $\{\gamma, \delta\}$ is red if γ, δ lie in the same X_α , and blue if they are in different X_α . Let $H \subseteq \kappa$ be a monochromatic subset of size κ for c . If H is red, then one of the X_α is large, which is a contradiction. But if H is blue, then λ must be large, which also gives a contradiction.

Suppose that κ is not a strong limit, so $2^\lambda \geq \kappa$ for $\lambda < \kappa$. Let B_λ be the set of functions $\lambda \rightarrow 2$, and give it the *lexicographic order*: we say that $f < g$ if $f(\alpha) < g(\alpha)$ at the first position α at which f

and g disagree. For this proof, we will use the combinatorial fact that this ordered structure $(B_\lambda, \leq_{\text{lex}})$ is a totally ordered set with no increasing or decreasing chains of length $\kappa > \lambda$. The proof is on an example sheet.

If $2^\lambda \geq \kappa$, there is a family of pairwise distinct elements $(f_\alpha)_{\alpha < \kappa}$ of B_λ of length κ . Define a colouring c of κ as follows. A pair α, β is red if the truth value of $\alpha \leq \beta$ is the same as the truth value of $f_\alpha \leq_{\text{lex}} f_\beta$. A pair is blue otherwise. Let H be a monochromatic set for c . If H is red, then f_α forms a \leq_{lex} -increasing sequence of length κ . If H is blue, then f_α forms a \leq_{lex} -decreasing sequence of length κ . Both results contradict the combinatorial result above. \square

Theorem. Every measurable cardinal is weakly compact.

Proof. Let $f : [\kappa]^2 \rightarrow 2$ be a colouring of a measurable cardinal κ . Let

$$X_0^\alpha = \{\beta \mid f(\{\alpha, \beta\}) = 0\}; \quad X_1^\alpha = \{\beta \mid f(\{\alpha, \beta\}) = 1\}$$

For a given α , these are disjoint, and $X_0^\alpha \cup X_1^\alpha = \kappa \setminus \{\alpha\}$, so precisely one of them lies in the ultrafilter U . Define $c : \kappa \rightarrow 2$ be such that $X_{c(\alpha)}^\alpha \in U$. Now, let

$$X_0 = \{\alpha \mid c(\alpha) = 0\}; \quad X_1 = \{\alpha \mid c(\alpha) = 1\}$$

Precisely one of these two sets lies in U .

We claim that if $X_i \in U$, then there is a monochromatic set H for colour i with $|H| = \kappa$. Without loss of generality, we may assume $i = 0$. Define

$$Z_\alpha = \begin{cases} X_0^\alpha & \text{if } c(\alpha) = 0 \\ \kappa & \text{if } c(\alpha) = 1 \end{cases}$$

Each of the Z_α lie in the ultrafilter U . As we may assume U is normal, the diagonal intersection of the Z_α also lies in U . So we can define

$$H = X_0 \cap \Delta_{\alpha < \kappa} Z_\alpha \in U$$

and $|H| = \kappa$. Let $\gamma < \delta$ with $\gamma, \delta \in H$. Then $\gamma, \delta \in X_0$, so $c(\gamma) = 0 = c(\delta)$. Hence $Z_\gamma = X_0^\gamma$ and $Z_\delta = X_0^\delta$. In particular,

$$\delta \in \Delta_{\alpha < \kappa} Z_\alpha \subseteq \bigcap_{\xi < \delta} Z_\xi \subseteq Z_\gamma = X_0^\gamma$$

Hence $f(\{\gamma, \delta\}) = 0$. \square

The large cardinal axioms discussed so far fall into a linear hierarchy of consistency strength. This is known as the *linearity phenomenon*.

2.5 Strongly compact cardinals

The compactness theorem for first-order logic says that for any first-order language L_S and set of axioms $\Phi \subseteq L_S$,

$$\Phi \text{ is satisfiable} \leftrightarrow (\forall \Phi_0 \subseteq \Phi. |\Phi_0| < \aleph_0 \rightarrow \Phi_0 \text{ is satisfiable})$$

This result cannot work for languages with infinitary conjunctions and disjunctions. Indeed, if we write

$$\varphi_F \equiv \bigvee_{i \in \mathbb{N}} \varphi_{=n}; \quad \varphi_{=n} \equiv \text{there are precisely } n \text{ elements}; \quad \varphi_{\geq n} \equiv \text{there are at least } n \text{ elements}$$

then

$$\{\varphi_{\geq n} \mid n \in \mathbb{N}\} \cup \{\varphi_F\}$$

is finitely satisfiable but not satisfiable.

Definition. An $\mathcal{L}_{\kappa\kappa}$ -language is defined by

- a set of variables;
- a set S of function, relation, and constant symbols of finite arity;
- the logical symbols $\wedge, \vee, \neg, \exists, \forall$; and
- the infinitary logical symbols $\bigwedge_{\alpha < \lambda}, \bigvee_{\alpha < \lambda}, \exists^\lambda, \forall^\lambda$ for $\lambda < \kappa$.

We define the new syntactic rules as follows. If φ_α are L_S -formulas for $\alpha < \lambda$, then so are $\bigwedge_{\alpha < \lambda} \varphi_\alpha$ and $\bigvee_{\alpha < \lambda} \varphi_\alpha$. If \mathbf{v} is a sequence of variables of length λ and φ is an L_S -formula, then $\exists^\lambda \mathbf{v}. \varphi$ and $\forall^\lambda \mathbf{v}. \varphi$ are L_S -formulas.

We say that M is a model of $\bigvee_{\alpha < \lambda} \varphi_\alpha$ if $M \models \varphi_\alpha$ for all $\alpha < \lambda$. Similarly, M models $\exists^\lambda \mathbf{v}. \varphi$ if there is a function $a : \lambda \rightarrow M$ such that

$$M \models \varphi \left[\begin{array}{c} a(0)a(1) \dots a(\xi) \dots \\ v_0 v_1 \dots v_\xi \dots \end{array} \right]$$

Definition. An $\mathcal{L}_{\kappa\kappa}$ -language L_S satisfies compactness if for all $\Phi \subseteq L_S$,

$$\Phi \text{ is satisfiable} \leftrightarrow (\forall \Phi_0 \subseteq \Phi. |\Phi_0| < \kappa \rightarrow \Phi_0 \text{ is satisfiable})$$

Note that if $\kappa = \omega$, we recover the standard notion of a first-order language, so all $\mathcal{L}_{\omega\omega}$ -languages satisfy compactness.

Definition. An uncountable cardinal κ is called *strongly compact*, denoted $\text{SC}(\kappa)$, if every $\mathcal{L}_{\kappa\kappa}$ -language satisfies compactness.

Theorem (Keisler–Tarski theorem). Suppose κ is a strongly compact cardinal. Then every κ -complete filter on κ can be extended to a κ -complete ultrafilter.

Proof. We define a language L extending the usual language of set theory by creating a constant symbol c_A for each $A \subseteq \kappa$, giving 2^κ -many symbols. Now let L^* be L with an extra constant symbol c . Let

$$M = (\mathcal{P}(\kappa), \in, \{A \mid A \subseteq \kappa\})$$

so c_A is interpreted by A . Let $\Phi = \text{Th}_L(M)$ be the L -theory of M . In particular,

$$M \models \forall x. x \in c_A \rightarrow x \text{ is an ordinal}$$

and

$$M \models \forall x. x \text{ is an ordinal} \rightarrow x \in c_A \vee x \in c_{\kappa \setminus A}$$

Now let

$$\Phi^* = \Phi \cup \{c \in c_A \mid A \in F\}$$

This is a subset of L^* . We show that Φ^* is κ -satisfiable. If $(A_\alpha)_{\alpha < \lambda}$ are subsets of κ such that $c \in c_{A_\alpha}$ occurs in a κ -small subset of Φ^* , then any element $\eta \in \bigcap_{\alpha < \lambda} A_\alpha$ can be chosen as the interpretation of c . As F is κ -complete, this intersection lies in F and so is nonempty as required.

Hence, by strong compactness of κ , the theory Φ^* is satisfiable. Let M be a model of Φ^* . Define

$$U = \{A \mid M \models c \in c_A\}$$

We claim that this is a κ -complete ultrafilter extending F . The fact that U extends F holds by construction of Φ^* . It is an ultrafilter because M believes that $c \in c_A$ or $c \in c_{\kappa \setminus A}$. It is κ -complete because if $\{A_\alpha \mid \alpha < \lambda\} \subseteq U$, let $A = \bigcap_{\alpha < \lambda} A_\alpha$, then

$$M \models \forall x. \left(x \in c_A \leftrightarrow \bigwedge_{\alpha < \lambda} x \in c_{A_\alpha} \right)$$

As this holds in particular for c , we obtain $A \in U$. □

Corollary. Every strongly compact cardinal is measurable.

Proof. Let

$$F = \{A \subseteq \kappa \mid |\kappa \setminus A| < \kappa\}$$

In the case $\kappa = \omega$, this is known as the Fréchet filter. This is a κ -complete filter on κ . If U extends F then U must be nonprincipal, so by the Keisler–Tarski theorem, F can be extended to a κ -complete nonprincipal ultrafilter on κ as required. □

3 Reflection

3.1 The Keisler extension property

Definition. A cardinal κ has the *Keisler extension property*, written $\text{KEP}(\kappa)$, if there is $\kappa \in X \supseteq V_\kappa$ transitive such that $V_\kappa \leq X$.

Proposition. If κ is inaccessible and satisfies the Keisler extension property, there is an inaccessible cardinal $\lambda < \kappa$.

Proof. Fix X as in the Keisler extension property. As κ is inaccessible, $X \models I(\kappa)$ because $\kappa \in X$ and inaccessibility is downwards absolute for transitive models. Also, $V_\kappa \models \text{ZFC}$, so $X \models \text{ZFC}$ as it is an elementary superstructure. Therefore, $X \models \text{ZFC} + \text{IC}$, so $V_\kappa \models \text{ZFC} + \text{IC}$. So as inaccessibility is absolute between V_κ and V , there is an inaccessible $\lambda < \kappa$. □

The phenomenon that properties of X occur below κ is called *reflection*. This argument can be improved in the following sense. For a given $\alpha < \kappa$,

$$X \models \exists \lambda > \alpha. I(\lambda)$$

But as $\alpha \in V_\kappa$, elementarity gives

$$V_\kappa \models \exists \lambda > \alpha. I(\lambda)$$

So the set

$$\{\lambda < \kappa \mid I(\lambda)\}$$

is not only nonempty, but cofinal in κ .

Corollary. Let A be the axiom

$$\exists \kappa. I(\kappa) \wedge \text{KEP}(\kappa)$$

Then

$$\text{ZFC} + \text{IC} <_{\text{Con}} \text{ZFC} + A$$

Proof. It suffices to show that $\text{ZFC} + A \models \text{Con}(\text{ZFC} + \text{IC})$. We have seen that $\text{ZFC} + A$ proves the existence of (at least) two inaccessible cardinals below κ , and in particular the larger of the two is a model of $\text{ZFC} + \text{IC}$. \square

Remark. This is the main technique for proving strict inequalities of consistency strength. Given two large cardinal properties Φ, Ψ with the appropriate amount of absoluteness properties, we show that $\text{ZFC} + \Phi(\kappa)$ proves that the set

$$\{\lambda < \kappa \mid \Psi(\lambda)\}$$

is cofinal in κ . Then $\text{ZFC} + \Phi \models \text{Con}(\text{ZFC} + \Psi)$.

Example. Consider the proof that every inaccessible cardinal has a worldly cardinal below it. In the construction, we produce a sequence of ordinals $(\alpha_i)_{i \in \omega}$, and the worldly cardinal is $\sup \alpha_i$. But we can set $\alpha_0 = \lambda + 1$ for a given worldly cardinal $\lambda < \kappa$, so this gives a cofinal sequence of worldly cardinals below every given inaccessible.

Theorem. Every strongly compact cardinal has the Keisler extension property.

Proof. We want to use the method of (elementary) diagrams to produce a model with V_κ as a substructure. However, we have no way to control whether such a model is well-founded using standard first-order model-theoretic techniques. To bypass this issue, we will use infinitary operators.

Let c_x be a constant symbol for each $x \in V_\kappa$, and let L be the language with \in and the c_x . Let

$$\mathcal{V} = (V_\kappa, \in, \{c_x \mid x \in V_\kappa\})$$

In first-order logic, $\text{Th}(X)$ is the elementary diagram of V_κ , so if $M \models \text{Th}(X)$, then $V_\kappa \subseteq M$. Let L_κ be the $\mathcal{L}_{\kappa\kappa}$ -language with the same symbols. Consider

$$\psi \equiv \forall^{\omega} \mathbf{v}. \bigvee_{i \in \omega} v_{i+1} \notin v_i$$

This expresses well-foundedness (assuming AC). Writing $\Phi = \text{Th}_{L_\kappa}(\mathcal{V})$ for the L_κ -theory of \mathcal{V} , we must have $\psi \in \Phi$ since V_κ is well-founded. Thus, if $M \models \Phi$, then M is a well-founded model containing V_κ . By taking the Mostowski collapse, we may also assume that any such M is transitive.

Extend L_κ to L_κ^+ with one extra constant c , and let

$$\Phi^+ = \Phi \cup \{c \text{ is an ordinal}\} \cup \{c \neq c_x \mid x \in V_\kappa\}$$

Any model of Φ^+ induces a transitive elementary superstructure of V_κ that contains an ordinal at least κ , so by transitivity, κ is in this model.

We show that Φ^+ is satisfiable by showing that it is κ -satisfiable, using the fact that κ is strongly compact. Let $\Phi^0 \subseteq \Phi^+$ be a subset of size less than κ . Then we can interpret c as some ordinal α greater than all ordinals β occurring in the sentences $c \neq c_\beta$ in Φ^+ . Then \mathcal{V} , together with this interpretation of c , is a model of Φ^0 . \square

Corollary.

$$\text{ZFC} + \text{IC} <_{\text{Con}} \text{ZFC} + \text{SCC}$$

The proof above only used languages with at most κ -many symbols. Let $\text{WC}(\kappa)$ be the axiom that every $\mathcal{L}_{\kappa\kappa}$ -language with at most κ -many symbols satisfies κ -compactness. Then we have shown that $\text{WC}(\kappa)$ implies the Keisler extension property. One can show that

$$\text{W}(\kappa) \leftrightarrow \text{WC}(\kappa)$$

So the cardinals κ that satisfy $\text{WC}(\kappa)$ are precisely the weakly compact cardinals. In particular,

$$\text{ZFC} + \text{IC} <_{\text{Con}} \text{ZFC} + \text{WCC}$$

Note that in the proof that strongly compact cardinals are measurable, we used a language with 2^κ -many symbols.

3.2 Ultrapowers of the universe

In order to avoid proper classes, we will consider ultrapowers of particular set universes. Later, we will briefly explain how all of this could have been done in a proper class universe such as V . For convenience, we will assume that $\kappa < \lambda$ where κ is measurable and λ is inaccessible, so $V_\lambda \models \text{ZFC} + \text{MC}$. We will take the ultrapower of V_λ .

Let U be a κ -complete nonprincipal ultrafilter on κ , and form the ultrapower of V_λ , consisting of equivalence classes of functions $f : \kappa \rightarrow V_\lambda$ where $f \sim g$ when $\{\alpha \mid f(\alpha) = g(\alpha)\} \in U$.

$$V_\lambda^\kappa / U = \{[f] \mid f : \kappa \rightarrow V_\lambda\}$$

The membership relation on the ultrapower is given by

$$[f] E [g] \leftrightarrow \{\alpha \mid f(\alpha) \in g(\alpha)\} \in U$$

We have an embedding ℓ from V_λ into the ultrapower by mapping $x \in V_\lambda$ to the equivalence class of its constant function $c_x : \kappa \rightarrow V_\lambda$. This is an elementary embedding by Łoś' theorem. Hence

$$(V_\lambda, \in) \equiv (V_\lambda^\kappa / U)$$

so they both model $\text{ZFC} + \text{MC}$, and in particular, $[c_\kappa]$ is a measurable cardinal.

Remark. (i) Suppose $V_\lambda^\kappa/U \models [f]$ is an ordinal. By Łoś' theorem,

$$X = \{\alpha \mid f(\alpha) \text{ is an ordinal}\} \in U$$

We can define

$$f'(\alpha) = \begin{cases} f(\alpha) & \text{if } \alpha \in X \\ 0 & \text{otherwise} \end{cases}$$

Note that $f \sim f'$, so $[f] = [f']$. So without loss of generality, we can assume f is a function into $\text{Ord} \cap \lambda = \lambda$, so $f : \kappa \rightarrow \lambda$. Since λ is inaccessible, f cannot be cofinal, so there is $\gamma < \lambda$ such that $f : \kappa \rightarrow \gamma$. Note also that, for example, we can define $f + 1$ by

$$(f + 1)(\alpha) = f(\alpha) + 1$$

so

$$\{\alpha \mid (f + 1)(\alpha) \text{ is the successor of } f(\alpha)\} = \kappa \in U$$

hence by Łoś' theorem, $[f + 1]$ is the successor of $[f]$.

(ii) If $f : \kappa \rightarrow V_\lambda$ is arbitrary, the set

$$\{\text{rank } f(\alpha) \mid \alpha \in \kappa\}$$

cannot be cofinal in λ , so there is $\gamma < \lambda$ such that $f \in V_\gamma$. However, the union of the equivalence class $[f]$ is unbounded in V_λ .

(iii) Given f , by (ii) we may assume $f \in V_\gamma$ for some $\gamma < \lambda$. If $[g] E [f]$, then

$$X = \{\alpha \mid g(\alpha) \in f(\alpha)\} \in U$$

Now we can define

$$g'(\alpha) = \begin{cases} g(\alpha) & \text{if } \alpha \in X \\ 0 & \text{otherwise} \end{cases}$$

Then $g \sim g'$ so $[g] = [g']$, and $g' \in V_\gamma$. Therefore,

$$|[g] \mid [g] E [f]| \leq |V_\gamma| < \lambda$$

Lemma. V_λ^κ/U is E -well-founded.

Proof. Suppose not, so let $\{[f_n] \mid n \in \mathbb{N}\}$ be a strictly decreasing sequence, so

$$[f_{n+1}] E [f_n]$$

By definition,

$$X_n = \{\alpha \mid f_{n+1}(\alpha) \in f_n(\alpha)\} \in U$$

But as U is κ -complete,

$$\bigcap_{n \in \mathbb{N}} X_n \in U$$

In particular, there must be an element $\alpha \in \bigcap_{n \in \mathbb{N}} X_n$. Hence, $f_n(\alpha)$ is an \in -decreasing sequence in V_λ , which is a contradiction. \square

Note that we only used \aleph_1 -completeness of U .

We can take the Mostowski collapse to produce a transitive set M such that

$$\pi : (\mathbb{V}_\lambda^\kappa / U, E) \cong (M, \in)$$

Combining ℓ and π , we obtain

$$j = \pi \circ \ell : (\mathbb{V}_\lambda, \in) \rightarrow (M, \in)$$

given by

$$j(x) = \pi(\ell(x)) = \pi([c_x])$$

For convenience, will write (f) to abbreviate $\pi([f])$, so $j(x) = (c_x)$.

Lemma. $M \subseteq \mathbb{V}_\lambda$.

Proof. Note that because λ is inaccessible, $\mathbb{V}_\lambda = H_\lambda$, where

$$H_\lambda = \{x \mid |\text{tcl}(x)| < \lambda\}$$

Since M is transitive, if $|x| < \lambda$ for each $x \in M$, then $M \subseteq H_\lambda$. But remark (iii) above shows precisely what is required. \square

Lemma. $\text{Ord} \cap M = \lambda$.

Proof. Under the elementary embedding j , ordinals in \mathbb{V}_λ are mapped to ordinals in M . So j restricts to an order-preserving embedding from λ into a subset of λ . Thus this embedding is unbounded, and therefore by transitivity, $\text{Ord} \cap M = \lambda$. \square

Lemma. $j|_{\mathbb{V}_\kappa} = \text{id}$, so in particular, $\mathbb{V}_\kappa \subseteq M$.

Proof. We show this by \in -induction on \mathbb{V}_κ . Suppose that $x \in \mathbb{V}_\kappa$ is such that for all $y \in x$, $j(y) = y$. For any $y \in x$, by elementarity, $j(y) \in j(x)$, but $j(y) = y$ so $y \in j(x)$ as required. For the converse, suppose $y \in j(x)$. Then define f such that $y = (f)$, so $(f) \in (c_x)$. Hence

$$X = \{\alpha \mid f(\alpha) \in c_x(\alpha)\} = \{\alpha \mid f(\alpha) \in x\} \in U$$

But

$$\{\alpha \mid f(\alpha) \in x\} = \bigcup_{z \in x} \{\alpha \mid f(\alpha) = z\}$$

This is a union of $|x|$ -many sets. By κ -completeness, there must be some $z \in x$ such that

$$\{\alpha \mid f(\alpha) = z\} \in U$$

Hence $f \sim c_z$. Therefore, $(f) = j(z)$, and by the inductive hypothesis, $j(z) = z$. Hence $y \in x$. \square

Lemma. $j \neq \text{id}$, as $j(\kappa) > \kappa$.

Proof. We know that $j(\kappa) = (c_\kappa)$. By the previous lemma, for each $\alpha < \kappa$, $j(\alpha) = (c_\alpha) = \alpha$. Consider the identity map $\text{id}_\kappa : \kappa \rightarrow \kappa$. We have

$$\begin{aligned} (c_\alpha) < (\text{id}_\kappa) &\leftrightarrow \{\gamma \mid c_\alpha(\gamma) < \text{id}_\kappa(\gamma)\} \in U \\ &\leftrightarrow \{\gamma \mid \alpha < \gamma\} \in U \end{aligned}$$

But by a size argument, $\{\gamma \mid \gamma \leq \alpha\} \notin U$ as U is nonprincipal, so we must have $\alpha < (\text{id})$. Also,

$$\begin{aligned} (\text{id}_\kappa) < (c_\kappa) &\leftrightarrow \{\gamma \mid \text{id}_\kappa(\gamma) < c_\kappa(\gamma)\} \in U \\ &\leftrightarrow \{\gamma \mid \gamma < \kappa\} \in U \end{aligned}$$

This is certainly in U . So for all $\alpha < \kappa$,

$$\alpha < (\text{id}_\kappa) < j(\kappa)$$

giving

$$\kappa \leq (\text{id}_\kappa) < j(\kappa)$$

as required. □

Remark. (i) This implies that $j|_{V_{\kappa+1}} \neq \text{id}$, so the identity result above cannot be strengthened.

(ii) This also shows that many of the elements of M arise from non-constant functions.

(iii) The set

$$\{j(x) \mid x \in V_\lambda\}$$

is isomorphic to V_λ . Therefore, there is a (non-transitive) copy of V_λ that sits strictly inside M .

(iv) Let $f : \kappa \rightarrow \kappa$ be a function such that for all $\gamma < \kappa$, $\text{id}_\kappa(\gamma) < f(\gamma)$. Then $(\text{id}_\kappa) < (f)$. For example, the functions $f_2(\gamma) = \gamma \cdot 2$ and $f_3(\gamma) = \gamma \cdot 3$ satisfy $(\text{id}_\kappa) < (f_2) < (f_3)$.

(v) At the moment, we do not know whether $(\text{id}_\kappa) = \kappa$. Consider

$$f(\gamma) = \begin{cases} \gamma - 1 & \text{if } \gamma \text{ is a successor} \\ \gamma & \text{if } \gamma \text{ is a limit} \end{cases}$$

Then

$$(f) < (\text{id}_\kappa) \leftrightarrow \{\alpha \mid \alpha \text{ is a limit}\} \notin U$$

We will discuss this in more detail later.

3.3 Properties above the critical point

Definition. Let $j : V_\lambda \rightarrow M$ be an elementary embedding such that $M \subseteq V_\lambda$ is transitive. An ordinal μ is called the *critical point* of j , written $\text{crit}(j)$, if $j \neq \text{id}$ and μ is the least ordinal α . such that $j(\alpha) > \alpha$.

Note that if $j \neq \text{id}$, it moves the rank of some set, so moves some ordinal. Therefore, if $j \neq \text{id}$, it has a critical point.

In this terminology, the critical point of the embedding j above is κ .

Remark. (i) M is closed under finite intersections: if $A, B \in M$, then $A \cap B \in M$.

(ii) $V_\kappa \in M$. To show this, we claim that the set

$$W = \{y \in M \mid M \models \text{rank } y < \kappa\}$$

is equal to V_κ . Then, since M models ZFC, the set W is V_κ^M , so $W \in M$.

If $x \in V_\kappa$, then $\text{rank } x = \alpha < \kappa$, so $j(x) = x$. By elementarity, $\text{rank } x = \text{rank } j(x) = j(\alpha) = \alpha$ as required. Conversely, suppose that $M \models \text{rank } y = \gamma$ for $\gamma < \kappa$. There is f such that $y = (f)$, and without loss of generality we can take $f : \kappa \rightarrow V_{\gamma+1}$. But $|V_{\gamma+1}| < \kappa$, and so by the argument in the lemma proving $j|_{V_\kappa} = \text{id}$, there is some $x \in V_{\gamma+1}$ such that $\{\alpha \mid f(\alpha) = x\} \in U$. Hence $f \sim c_x$, and so $y = j(x) = x$.

Lemma. $V_{\kappa+1} \subseteq M$.

Note that $j|_{V_{\kappa+1}} \neq \text{id}$.

Proof. Let $A \in V_{\kappa+1}$, so $A \subseteq V_\kappa$. We claim that $A = j(A) \cap V_\kappa$. Then, by the two remarks above, this implies $A \in M$.

Suppose $x \in A \subseteq V_\kappa$. By elementarity, $j(x) \in j(A)$, but $x = j(x)$, so $x \in j(A)$. Conversely, suppose $x \in j(A) \cap V_\kappa$. Then $x = j(x)$, so $j(x) \in j(A)$. So by elementarity in the other direction, $x \in A$. \square

Lemma. $V_\lambda \models |j(\kappa)| \leq 2^\kappa$.

Proof. Recall that if $f \in V_\gamma$ then $|f| \leq |V_\gamma|$. So if $(f) \in j(\kappa) = (c_\kappa)$, we can assume $f : \kappa \rightarrow \kappa$, and there are only 2^κ -many such functions. \square

In particular, V_λ believes that $j(\kappa)$ is not a strong limit cardinal. Hence,

Lemma. $M \neq V_\lambda$.

Proof. M believes that $j(\kappa)$ is measurable, so in particular it believes $j(\kappa)$ is a strong limit. Hence $M \neq V_\lambda$. \square

There is a strengthening of this result which exhibits a witness to $M \subsetneq V_\lambda$, discussed on the example sheets. Namely, we can show that $U \notin M$. In order to show this, we prove that for arbitrary transitive $N \subseteq V_\lambda$ with $U \in N$, we have $N \models |j(\kappa)| \leq 2^\kappa$. In particular, $V_{\kappa+2} \not\subseteq M$.

Note that M might still believe that κ is measurable, even though $U \notin M$. There could be some other $U' \in V_{\kappa+2}$ which is κ -complete and nonprincipal.

Recall that the Keisler extension property for a transitive model X is the statement that there is $\kappa \in X$ such that $V_\kappa \leq X$. Properties of X reflect down into V_κ : if $\alpha \in \text{Ord}^{V_\kappa}$ and Φ is a property such that $X \models \Phi(\kappa)$, then

$$X \models \exists \mu. \alpha < \mu \wedge \Phi(\mu)$$

so

$$V_\kappa \models \exists \mu. \alpha < \mu \wedge \Phi(\mu)$$

hence

$$C_\Phi = \{\gamma < \kappa \mid \Phi(\gamma)\} \subseteq \kappa$$

is cofinal in κ . Now, if Φ is any property such that $M \models \Phi(\kappa)$, then for any $\alpha < \kappa$,

$$M \models \exists \mu. j(\alpha) < \mu < j(\kappa) \wedge \Phi(\mu)$$

By elementarity,

$$V_\lambda \models \exists \mu. \alpha < \mu < \kappa \wedge \Phi(\mu)$$

Note that $\alpha = j(\alpha)$. So

$$C_\Phi = \{\gamma < \kappa \mid \Phi(\gamma)\}$$

is cofinal in κ .

Example. (i) Let $\Phi(\kappa) = I(\kappa)$ be the statement that κ is inaccessible. By absoluteness, $M \models I(\kappa)$, so

$$C_I = \{\gamma < \kappa \mid I(\gamma)\}$$

is cofinal. So if κ is measurable, it is the κ th inaccessible cardinal.

(ii) Let $\Phi(\kappa) = W(\kappa)$ be the statement that κ is weakly compact. We show that $M \models W(\kappa)$. Let $c : [\kappa]^2 \rightarrow 2$ be a colouring in M ; we find $H \in [\kappa]^\kappa$ in M that is monochromatic for c . By the fact that $V_\lambda \models W(\kappa)$, we obtain H as above in V_λ . But this H is a subset of κ , so is an element of $V_{\kappa+1} \subseteq M$ as required. By the reflection argument,

$$C_W = \{\gamma < \kappa \mid W(\gamma)\}$$

is cofinal in κ . So the least weakly compact cardinal is not measurable.

Definition. A property Φ is called β -stable if for all transitive models M and all κ , if $\Phi(\kappa)$ holds and $V_{\kappa+\beta} \subseteq M$ then $M \models \Phi(\kappa)$.

Remark. (i) Weak compactness is 1-stable, and 1-stable properties of measurable cardinals reflect at a measurable cardinal.

(ii) Measurability is 2-stable, because the property Ξ of being a κ -complete nonprincipal ultrafilter is absolute, but the existence of the ultrafilter requires two power set operations:

$$M(\kappa) \leftrightarrow \exists U \in V_{\kappa+2}. \Xi(U)$$

Example. Suppose that $M \models M(\kappa)$. Then by the same reflection argument, the set C_M is cofinal in κ , so κ is the κ th measurable cardinal, and so is not the least.

Definition. A cardinal κ is called *surviving*, written $\text{Surv}(\kappa)$, if there is $\lambda > \kappa$ inaccessible, a κ -complete nonprincipal ultrafilter on κ , a transitive model M such that $M \cong V_\lambda^\kappa / U$ and j is the elementary embedding derived from U , where $M \models M(\kappa)$.

By the example above, if κ is the first surviving cardinal, it is the κ th measurable. Under sufficient consistency assumptions, we have the following.

Corollary. $MC <_{\text{Con}} \text{SurvC}$.

Proof. Let κ be a surviving cardinal. By the previous results, we can find $\lambda_0 < \lambda_1 < \kappa$ such that λ_0, λ_1 are both measurable. Then λ_1 is inaccessible, so $V_{\lambda_1} \models \text{ZFC} + M(\lambda_0)$ by 2-stability of measurability and the fact that $V_{\lambda_0+2} \subseteq V_{\lambda_1}$. \square

3.4 The fundamental theorem on measurable cardinals

Theorem. Suppose λ is inaccessible and $\kappa < \lambda$. Then the following are equivalent.

- (i) κ is measurable.
- (ii) There is a transitive model M of ZFC with $V_{\kappa+1} \subseteq M$ and an elementary embedding $j : V_\lambda \rightarrow M$ such that $j \neq \text{id}$ and $\kappa = \text{crit}(j)$.

Proof. We have already shown that (i) implies (ii). For the converse, we define an ultrafilter U by

$$U = \{A \subseteq \kappa \mid \kappa \in j(A)\}$$

Note that if $A \subseteq \kappa$, then $j(A) \subseteq j(\kappa)$, so it could in fact be the case that $\kappa \in j(A)$. We show that U is a κ -complete nonprincipal ultrafilter.

- We have $\kappa \in U$ precisely if $\kappa \in j(\kappa)$, but this is true as κ is the critical point of j .
- $\emptyset \in U$ precisely if $\kappa \in j(\emptyset)$, but $j(\emptyset) = \emptyset$ as j is an elementary embedding.
- If $A \in U$ and $B \supseteq A$, then $\kappa \in j(A)$, but $j(B) \supseteq j(A)$ by elementarity, so $\kappa \in j(B)$ giving $B \in U$.
- Suppose $A \notin U$. Then $\kappa \notin j(A)$. We want to show $\kappa \setminus A \in U$, or equivalently, $\kappa \in j(\kappa \setminus A)$. By elementarity, $j(\kappa \setminus A) = j(\kappa) \setminus j(A)$. But $\kappa \in j(\kappa) \setminus j(A)$ as required.
- We show U is nonprincipal. Let $\alpha \in \kappa$. Then $\{\alpha\} \in U$ precisely when $\kappa \in j(\{\alpha\}) = \{j(\alpha)\}$. But $\alpha < \kappa$, so $j(\alpha) = \alpha \neq \kappa$, hence U cannot be principal.
- Finally, we show κ -completeness; this will also show the finite intersection property required for U to be a filter. Let $\gamma < \kappa$, and fix $(A_\alpha)_{\alpha < \gamma}$ such that $A_\alpha \in U$ for each $\alpha < \gamma$. Then $\kappa \in j(A_\alpha)$ for all $\alpha < \gamma$. Then $\bigcap_{\alpha < \gamma} A_\alpha \in U$ if and only if $\kappa \in j\left(\bigcap_{\alpha < \gamma} A_\alpha\right)$. Note that being an element of $\bigcap_{\alpha < \gamma} A_\alpha$ is a formula that says that \mathbf{A} is a sequence of objects A_α , the α th element of this sequence is A_α , and β is an element of each element of the sequence. Therefore $\beta \in j\left(\bigcap_{\alpha < \gamma} A_\alpha\right)$ if and only if β is an element of all elements of the sequence $j(\mathbf{A})$. Clearly, $j(\mathbf{A})$ is a sequence of subsets of $j(\kappa)$ of length $j(\gamma) = \gamma$. Since A_α is the α th element of \mathbf{A} , $j(A_\alpha)$ is the $j(\alpha)$ th element of $j(\mathbf{A})$, but $j(\alpha) = \alpha$. Hence $j(\mathbf{A})$ is the sequence $(j(A_\alpha))_{\alpha < \gamma}$. Then

$$j\left(\bigcap_{\alpha < \gamma} A_\alpha\right) = \bigcap_{\alpha < \gamma} j(A_\alpha)$$

giving κ -completeness as required. \square

Remark. Given a sequence \mathbf{A} of subsets of κ of length γ , then $j(\mathbf{A})$ is a sequence of subsets of $j(\kappa)$ of length $j(\gamma)$. Moreover, if A_α is the α th element of \mathbf{A} , then $j(A_\alpha)$ is the $j(\alpha)$ th element of $j(\mathbf{A})$. In the situation above, $\gamma < \kappa$, so $j(\gamma) = \gamma$ and $j(\alpha) = \alpha$, so $j(\mathbf{A}) = \{j(A_\alpha) \mid \alpha < \gamma\}$. If, for example, $\gamma = \kappa$, then $j(\mathbf{A})$ is a sequence of length $j(\kappa)$, which is strictly longer. Despite this, the first κ -many elements of the sequence are still $j(A_\alpha)$ for $\alpha < \kappa$. Beyond κ , we do not know what the elements of $j(\mathbf{A})$ look like. This remark suffices for the following result.

Proposition. For arbitrary embeddings j with critical point κ , the ultrafilter U_j constructed above is normal.

Proof. Suppose $A_\alpha \in U_j$ for each $\alpha < \kappa$, or equivalently, $\kappa \in j(A_\alpha)$. We must show $\kappa \in j(\Delta_{\alpha < \kappa}(A_\alpha))$. We have

$$\begin{aligned} \xi \in \Delta_{\alpha < \kappa}(A_\alpha) &\leftrightarrow \xi \in \bigcap_{\alpha < \xi} A_\alpha \\ &\leftrightarrow \forall \alpha < \xi. \xi \in A_\alpha \\ \xi \in j\left(\Delta_{\alpha < \kappa}(A_\alpha)\right) &\leftrightarrow \forall \alpha < \xi. \xi \in j(\mathbf{A})_{j(\alpha)} \end{aligned}$$

Substitute κ for ξ and obtain

$$\begin{aligned} \kappa \in j\left(\Delta_{\alpha < \kappa}(A_\alpha)\right) &\leftrightarrow \forall \alpha < \kappa. \kappa \in j(\mathbf{A})_{j(\alpha)} \\ &\leftrightarrow \forall \alpha < \kappa. \kappa \in j(\mathbf{A})_\alpha \\ &\leftrightarrow \forall \alpha < \kappa. \kappa \in j(A_\alpha) \end{aligned}$$

which holds by assumption. □

Remark. (i) This gives an alternative proof of the existence of a normal ultrafilter on a measurable cardinal.

(ii) The operations $U \mapsto j_U$ and $j \mapsto U_j$ are not inverses in general. In particular, if U is not normal, $U_{j_U} \neq U$.

Proposition. Let U be a κ -complete nonprincipal ultrafilter on κ . Then the following are equivalent.

- (i) U is normal;
- (ii) $(\text{id}) = \kappa$.

This proposition provides an alternative view of reflection. Suppose that the ultrafilter U on κ is normal. If $M \models \Phi(\kappa)$, then $M \models \Phi((\text{id}))$. By Łoś' theorem,

$$\{\alpha < \kappa \mid \Phi(\text{id}(\alpha))\} \in U$$

So Φ reflects not only on a set of size κ , but on an ultrafilter set. In particular, if $\Phi = M$ and $M \models M(\kappa)$, so if κ is surviving, then the set of α that are measurable is in U . Using this result, we can characterise the surviving cardinals in a more elegant way.

Theorem. κ is surviving if and only if there is a normal ultrafilter on κ such that $\{\alpha < \kappa \mid M(\alpha)\} \in U$.

Proof. We have just shown one direction. For the converse, suppose the set $C = \{\alpha < \kappa \mid M(\alpha)\}$ is in U . Then for each $\alpha \in C$, one can find an α -complete nonprincipal ultrafilter on α called U_α . Define

$$f(\alpha) = \begin{cases} U_\alpha & \text{if } \alpha \in C \\ \emptyset & \text{if } \alpha \notin C \end{cases}$$

Thus the set of α such that $f(\alpha)$ is an α -complete nonprincipal ultrafilter on α is C , so in U . Equivalently, the set of α such that $f(\alpha)$ is an $\text{id}(\alpha)$ -complete nonprincipal ultrafilter on $\text{id}(\alpha)$ is in U . So by Łoś' theorem, M believes that (f) is an (id) -complete nonprincipal ultrafilter on (id) . So (f) witnesses that κ is measurable in M . \square

This shows that whether a cardinal κ is surviving depends only on $V_{\kappa+2}$, and is therefore a 2-stable property.

Definition. If U, U' are normal ultrafilters on κ , we write $U <_M U'$ if

$$C = \{\alpha \mid M(\alpha)\} \in U$$

and there is a sequence of ultrafilters U_α on $\alpha \in C$ such that

$$A \in U' \leftrightarrow \{\alpha \mid A \cap \alpha \in U_\alpha\} \in U$$

This is known as the *Mitchell order*.

Then κ is surviving if and only if there are U, U' on κ such that $U <_M U'$, because of the fact that if $h(\alpha) = A \cap \alpha$ then $(h) = A$. Note that talking about sequences of Mitchell-ordered ultrafilters is also 2-stable.

4 Towards inconsistency

4.1 Strong cardinals

Definition. A large cardinal axiom ΦC is called an *embedding axiom* if $\Phi(\kappa)$ holds if and only if there is a transitive model M and elementary embedding $j : V_\lambda \rightarrow M$ with critical point κ with certain additional properties.

$M(\kappa)$ is the simplest embedding axiom. The remaining large cardinal axioms in this course will take the form of embedding axioms.

Definition. An embedding $j : V_\lambda \rightarrow M$ with critical point κ is called β -strong if $V_{\kappa+\beta} \subseteq M$. A cardinal κ is called β -strong if there is a β -strong embedding with critical point κ .

β -stable properties are preserved by β -strong embeddings. In particular, by the reflection argument, if Φ is β -stable and κ is β -strong with $\Phi(\kappa)$, then κ is the κ th cardinal with property Φ .

Note that κ is measurable if and only if κ is 1-strong, and if κ is 2-strong then $\{\alpha < \kappa \mid M(\alpha)\}$ and $\{\alpha < \kappa \mid \text{Surv}(\alpha)\}$ are of size κ . If we write $\beta\text{-S}(\kappa)$ to denote that κ is β -strong, then

$$\text{SurvC} <_{\text{Con}} 2\text{-S}(\kappa)$$

This also gives an example of $j_{U_j} \neq j$, as the ultrapower embedding of any ultrafilter is never 2-strong.

Definition. A large cardinal property Φ is said to have *witness objects* of rank β if there is a formula Ψ that is downwards absolute for transitive models such that

$$\Phi(\kappa) \leftrightarrow \forall x. \exists y \in V_{\kappa+\beta}. \Psi(x, y, \kappa)$$

Any large cardinal property with witness objects of rank β is β -stable.

Example. (i) Weakly compact cardinals have witness objects of rank 1: for all colourings, there exists a homogeneous set in $V_{\kappa+1}$.

(ii) Measurable cardinals have witness objects of rank 2: there is a κ -complete nonprincipal ultrafilter on κ . The initial $\forall x$ quantifier is not needed in this case.

(iii) Surviving cardinals also have witness objects of rank 2, namely, a pair of ultrafilters.

In particular, inaccessibility is 0-stable, weak compactness is 1-stable, and measurability and survivability are 2-stable.

Remark. If β -strong cardinals have witness objects, they cannot be of rank β , because then they would reflect below. Witness objects for strength exist and are called *extenders*, and if μ is the least \beth fixed point larger than $|V_{\kappa+\beta}|$, then the witness object for β -strength has rank at most μ .

Definition. A cardinal κ is called *strong* if it is β -strong for all $\beta < \lambda$.

Importantly, the quantifiers are

$$\forall \beta. \exists j. V_{\kappa+\beta} \subseteq M$$

This does not say that there exists an embedding where all of the $V_{\kappa+\beta}$ are subsets of the same M . This notion cannot have a single witness object of a fixed rank, since otherwise, strength would reflect strength.

4.2 Removing the inaccessible

The ultrapower constructions used an inaccessible cardinal above a measurable cardinal, so that we could obtain a set-sized universe containing a measurable cardinal. When trying to do this with the real universe, we encounter several problems.

(i) The definition of ultrapowers requires a set model.

(ii) In the fundamental theorem of measurable cardinals, we have a quantification over j and M . If these are proper classes, this quantification cannot be expressed in the usual language of set theory.

(iii) Also, in the fundamental theorem of measurable cardinals, we use the notion of an elementary embedding, which is only definable for set models.

To solve problem (i), we would like to construct V^κ / \sim_U . Note that V^κ is a well-defined class; it is the class of all functions with domain κ . For such functions, it is easy to define the equivalence relation \sim_U . However, the equivalence classes $[f]_U = \{g \in V^\kappa \mid f \sim_U g\}$ are all proper classes. So V^κ / \sim_U is no longer a standard class; classes containing proper classes are typically not allowed. This can be resolved using *Scott's trick*. If C is a nonempty class, then there is a minimal α such that $C \cap V_\alpha \neq \emptyset$. This is a nonempty set. Define $\text{scott}(C) = C \cap V_\alpha$ for this α . Hence, if $[f]_U \neq [g]_U$, we have $\text{scott}([f]_U) \neq \text{scott}([g]_U)$. We can therefore define

$$V^\kappa / \sim_U = \{\text{scott}([f]_U) \mid \text{dom } f = \kappa\}$$

To obtain our model M , we took the Mostowski collapse of V^κ / \sim_U . Therefore, we need a class version of the Mostowski collapse. Recall that a relation $E \subseteq C \times C$ is *set-like* if for all $x \in C$, the class $\{y \in C \mid y E x\}$ is a set.

Theorem. Let C be a class, and let $E \subseteq C \times C$ be a binary relation on C that is well-founded, extensional, and set-like. Then there is a unique transitive class T such that $(T, \in) \cong (C, E)$.

This may be proven in an almost identical fashion to Mostowski's collapsing theorem for sets.

For problems (ii) and (iii), recall that the fundamental theorem of measurable cardinals was that $M(\kappa)$ is equivalent to the statement that there is an elementary embedding $j : V_\lambda \rightarrow M$ with critical point κ . Measurability is witnessed at $\kappa + 2$, but the elementary embedding is not witnessed anywhere below λ , so we cannot extend this definition to the usual universe. We can solve this by extending our set theory to an appropriate class theory. Standard class theories include *von Neumann–Bernays–Gödel* or NBG, and *Morse–Kelley* or MK. These theories have very different notions of class. NBG set theory is based upon the idea that definable formulas give the classes. It is a 'minimal class theory' where all classes are definable. MK is based on the idea that Ord behaves externally like an inaccessible cardinal. In this theory, there could be undefinable classes, and more classes than sets.

This resolves problem (ii), as we are permitted to work in a language in which we may quantify over proper classes. However, this does not solve problem (iii). Elementarity cannot be expressed as a single formula, but becomes a schema. This causes additional problems as we need the existential over j and M to be part of each formula. This could be solved by extending the language to add symbols for j and M . Another resolution is to observe that Σ_1 -elementarity suffices, as is explored in Kanamori's book *The Higher Infinite* on page 45. This can be defined using a single formula, therefore solving problem (iii).

4.3 Supercompact cardinals

Definition. M is closed under μ -sequences if $M^\mu \subseteq M$.

Theorem. If κ is measurable and $j : V_\lambda \rightarrow M$ is the ultrapower embedding, then M is closed under κ -sequences but not κ^+ -sequences.

Proof. Let $S = \{(f_\alpha) \mid \alpha < \kappa\} \in M^\kappa$. We must show that $S \in M$. Find h such that $(h) = \kappa$. For $\xi \in \kappa$, define $g(\xi)$ to be a function with domain $h(\xi)$ such that for all $\alpha \in h(\xi)$,

$$g(\xi)(\alpha) = f_\alpha(\xi)$$

Then

$$\{\xi \mid \text{dom } g(\xi) = h(\xi)\} = \kappa \in U$$

By Łoś' theorem, $\text{dom}(g) = (h) = \kappa$. Further,

$$\{\xi \mid \forall \alpha \in \text{dom } g(\xi). g(\xi)(\alpha) = f_\alpha(\xi)\} = \kappa \in U$$

so again by Łoś' theorem, if $\alpha \in \text{dom}(g) = \kappa$, then $(g)(\alpha) = (f_\alpha)$. Hence $(g) = S$.

Let

$$T = \{j(\alpha) \mid \alpha < \kappa^+\} \in M^{\kappa^+}$$

We claim that $T \notin M$. To prove this, we first show that T is unbounded in $j(\kappa^+)$, which is equal to $j(\kappa)^+$ by elementarity. Indeed, consider an arbitrary $(f) < j(\kappa^+)$. Then $j(\kappa^+) = (c_{\kappa^+})$, so without loss of generality we can assume $f : \kappa \rightarrow \kappa^+$. As κ^+ is regular, f is bounded by some $\alpha < \kappa^+$, so $f : \kappa \rightarrow \alpha$. Then $(f) < (c_\alpha) = j(\alpha) \in T$.

Now, note that $j(\kappa^+) = j(\kappa)^+$ is a regular cardinal, so cannot have small unbounded subsets. But $|T| = \kappa^+ < j(\kappa)^+$, so $T \notin M$. \square

Definition. An embedding j is called μ -supercompact if $M^\mu \subseteq M$. A cardinal κ is called μ -supercompact if there is a μ -supercompact embedding with critical point κ .

Therefore, the theorem above shows that if κ is measurable, then it is κ -supercompact, and the ultra-power embedding is not κ^+ -supercompact, although there could be other embeddings that are.

Definition. A cardinal κ is called *supercompact* if it is μ -supercompact for all $\mu < \lambda$.

As with strong cardinals, the quantifiers are in the order

$$\forall \mu. \exists j. M^\mu \subseteq M$$

If κ is 2^κ -supercompact, then κ is 2-strong. First note that $V_{\kappa+2} = \mathcal{P}(V_{\kappa+1})$ and $|V_{\kappa+1}| = |\mathcal{P}(V_\kappa)| = 2^\kappa$. Every $A \in V_{\kappa+1}$ is a 2^κ -sequence of elements of M , so if every 2^κ -length sequence lies in M , then $A \in M$ as required. In general, if κ is $|V_{\kappa+\beta}| = \beth_{\kappa+\beta}$ -supercompact, then κ is $(\beta + 1)$ -strong. In particular,

Corollary. Every supercompact cardinal is strong.

4.4 The upper limit

We now consider reversing the quantifier order in the definition of a strong cardinal.

Definition. A cardinal κ is called *Reinhardt* if there is an embedding j such that for all β , we have $V_{\kappa+\beta} \subseteq M$, or equivalently, $M = V_\lambda$. In other words, there is an elementary embedding $j : V_\lambda \rightarrow V_\lambda$ with critical point κ .

Theorem (Kunen). ZFC proves that there are no Reinhardt cardinals.

It is an open problem whether ZF without AC proves there are no Reinhardt cardinals.

Proof. Suppose $j : V_\lambda \rightarrow M$ has critical point κ . Find the least j -fixed point above κ , by defining

$$\kappa_0 = \kappa; \quad \kappa_{i+1} = j(\kappa_i); \quad \hat{\kappa} = \bigcup_{i \in \omega} \kappa_i$$

so $j(\hat{\kappa}) = \hat{\kappa}$. We will show that $V_{\hat{\kappa}+1} \not\subseteq M$, which is a result called *Kunen's lemma*. This contradicts the assumption that $M = V_\lambda$.

We need a combinatorial lemma due to Erdős and Hajnal. For a cardinal δ , we say that $f : [\delta]^\omega \rightarrow \delta$ is ω -Jónsson if for every $X \subseteq \delta$ such that $|X| = \delta$, we have $\{f(A) \mid A \in [X]^\omega\} = \delta$. The lemma states that every cardinal δ has an ω -Jónsson function.

Suppose that $V_{\hat{\kappa}+1} \subseteq M$. Let $f : [\hat{\kappa}]^\omega \rightarrow \hat{\kappa}$ be an ω -Jónsson function for $\hat{\kappa}$. Then M believes that $j(f)$ is an ω -Jónsson function for $j(\hat{\kappa}) = \hat{\kappa}$. Define

$$X = \{j(\alpha) \mid \alpha \in \hat{\kappa}\} \in V_{\hat{\kappa}+1}$$

We claim that $X \notin M$, finishing the proof. Suppose $X \in M$. Clearly $|X| = \hat{\kappa}$, so then $M \models |X| = \hat{\kappa}$. We can apply the definition of an ω -Jónsson function in M , which shows that

$$M \models \{j(f)(A) \mid A \in [X]^\omega\} = \hat{\kappa}$$

Any $A \in [X]^\omega$ is of the form $\{j(\alpha_i) \mid i \in \omega\}$ for some $a = \{\alpha_i \mid i \in \omega\} \in [\hat{\kappa}]^\omega$. Then $j(a) = \{j(\alpha_i) \mid i \in \omega\} = A$.

In general, if $g(x) = y$, then g is a function, $x \in \text{dom } g$, and $\langle x, y \rangle \in g$. Applying j , we have that $j(g)$ is a function, $j(x) \in \text{dom } j(g)$, and $\langle j(x), j(y) \rangle \in j(g)$. So $j(g)(j(x)) = j(y) = j(g(x))$. Therefore, we obtain $j(f)(A) = j(f)(j(a)) = j(f(a))$. But $f(a) \in \hat{\kappa}$, so $j(f(a)) \in X$. Therefore,

$$M \models \hat{\kappa} = \{j(f)(A) \mid A \in [X]^\omega\} \subseteq X$$

But this cannot be true, for example because $\kappa \notin X$ but $\kappa \in \hat{\kappa}$. □

Remark. (i) The combinatorial lemma was proven using AC, and it is not known whether the proof works without it.

(ii) To prove Kunen's lemma, we did not need that λ is inaccessible. More explicitly, if $j : V_\alpha \rightarrow M$ is an elementary embedding with critical point κ such that $\hat{\kappa} + 2 \leq \alpha$ (to guarantee that $f \in V_\alpha$), then $V_{\hat{\kappa}+1} \not\subseteq M$.

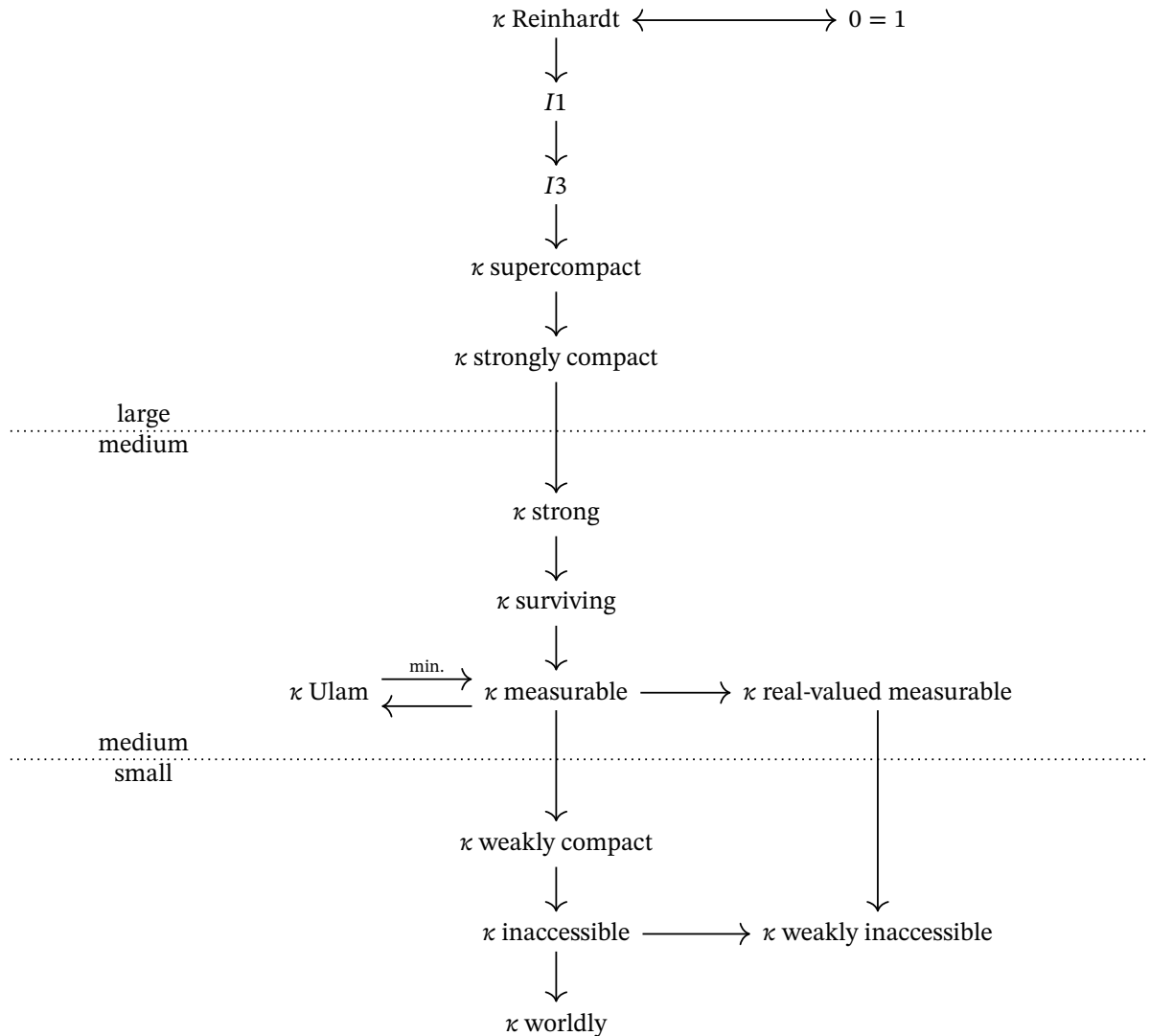
Corollary. For any ordinal δ , there is no elementary embedding $j : V_{\delta+2} \rightarrow V_{\delta+2}$ with critical point less than $\delta + 2$.

Proof. Observe that if $\kappa < \delta + 2$ is the critical point, we cannot have $\kappa = \delta$ or $\kappa = \delta + 1$, because δ and $\delta + 1$ are definable in $V_{\delta+2}$. Then $j(\kappa) < \delta$, so by induction all of the iterated images of κ under j are less than δ , so $\hat{\kappa} \leq \delta$. Thus $\hat{\kappa} + 2 \leq \delta + 2$, so by remark (ii), $V_{\hat{\kappa}+1} \not\subseteq V_{\delta+2}$, giving a contradiction. \square

The axiom stating the existence of an analogous $j : V_{\delta+1} \rightarrow V_{\delta+1}$ is called *I1*, and the existence of $j : V_{\delta} \rightarrow V_{\delta}$ is called *I3*; there is an axiom *I2* in between. Clearly, if $j : V_{\delta+1} \rightarrow V_{\delta+1}$ is elementary, then so is $j|_{V_{\delta}} : V_{\delta} \rightarrow V_{\delta}$, so *I1* implies *I3*. It has been hypothesised that *I1* and *I3* are inconsistent, but we do not yet have a proof.

Diagram of large cardinal properties

Under suitable consistency assumptions, large cardinal properties that appear in higher positions on this diagram have strictly higher consistency strength than properties appearing lower down the diagram.



The ‘small large cardinals’ are usually considered those cardinals consistent with $V = L$, and such large cardinal properties are usually downwards absolute. Note that L has no measurable cardinals. Indeed, if $V = L$ and U is a κ -complete nonprincipal ultrafilter on κ , then the ultrapower embedding $j_U : L \rightarrow M$ must map to an inner model strictly smaller than L , but such an inner model cannot exist.

There are certain large cardinals called *Woodin cardinals* which sit between strong and strongly compact cardinals. They represent another boundary between sizes of large cardinal axioms, just like

measurable cardinals; smaller large cardinals are sometimes called ‘medium-sized large cardinals’, and the others are called ‘large large cardinals’. Woodin cardinals are crucial for understanding the connection between large cardinals and infinite games. We know very little about large cardinal axioms beyond Woodin cardinals.