Group Cohomology

Cambridge University Mathematical Tripos: Part III

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Contents

1 Definitions and resolutions

1.1 ???

Let G be a group.

Definition. The *integral group ring* $\mathbb{Z}G$ is the set of formal sums $\sum n_g g$, where $n_g \in \mathbb{Z}$, $g \in G$, and only finitely many of the n_g are nonzero. An addition operation makes this set a free abelian group:

$$\left(\sum m_g g\right) + \left(\sum n_g g\right) = \sum (m_g + n_g) g$$

Multiplication is defined by

$$\left(\sum_{h \in G} m_h h\right)\left(\sum_{k \in G} n_k k\right) = \sum \left(\sum_{hk = g} m_h n_k\right) g$$

The multiplicative identity is $1e$ where e is the identity of G. This produces an associative ring, which underlies the integral representation theory of G.

Definition. A *(left) $\mathbb{Z}G$-module M* is an abelian group under addition together with a map $\mathbb{Z}G \times M \to M$ denoted $(r, m) \mapsto rm$, satisfying

(i) $r(m_1 + m_2) = rm_1 + rm_2$;
(ii) $(r_1 + r_2)m = r_1 m + r_2 m$;
(iii) $r_1(r_2 m) = (r_1 r_2)m$;
(iv) $1m = m$.

A module is *trivial* if $gm = m$ for all $g \in G$ and $m \in M$. We call $\mathbb{Z}$ *the* trivial module, given by the trivial action $gn = n$ for all $n \in \mathbb{Z}$ and $g \in G$.

The *free* $\mathbb{Z}G$-module on a set X is the module of formal sums $\sum r_x x$ where $r_x \in \mathbb{Z}G$ and $x \in X$, and only finitely many of the r_x are nonzero. This has the obvious G-action. This module will be denoted $\mathbb{Z}G\{X\}$.

We can define submodules, quotient modules, and so on as one would expect.

Definition. A *(left) $\mathbb{Z}G$-map* or *morphism* $\alpha : M_1 \to M_2$ is a map of abelian groups with $\alpha(rm) = r\alpha(m)$ for all $r \in \mathbb{Z}G$ and $m \in M_1$.

Example. The *augmentation map* $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ is the $\mathbb{Z}G$-map between left $\mathbb{Z}G$-modules given by

$$\sum n_g g \mapsto \sum n_g$$

This is also a right $\mathbb{Z}G$-map, and also a map of rings.

We will write $\mathrm{Hom}_G(M, N)$ to be the set of $\mathbb{Z}G$-maps $M \to N$, which is made into an abelian group under pointwise addition.

Example. Regarding $\mathbb{Z}G$ as a left $\mathbb{Z}G$-module, then

$$\mathrm{Hom}_G(\mathbb{Z}G, M) \cong M$$

for any left $\mathbb{Z}G$-module M. This isomorphism is given by $\varphi \mapsto \varphi(1)$; the $\mathbb{Z}G$-map is determined by the image of 1.

$$\varphi(r) = \varphi(r \cdot 1) = r\varphi(1)$$

Note that $\mathrm{Hom}_G(\mathbb{Z}G, M)$ can be viewed as a left $\mathbb{Z}G$-module, given by

$$(s\varphi)(r) = \varphi(rs); \quad s \in \mathbb{Z}G$$

Note that the isomorphism

$$\mathrm{Hom}_G(\mathbb{Z}G, \mathbb{Z}G) \cong \mathbb{Z}G; \quad \varphi \mapsto \varphi(1)$$

satisfies $\varphi(r) = r\varphi(1)$ and so φ corresponds to multiplication on the right by $\varphi(1)$.

Remark. G may not be abelian, and so we must carefully distinguish left and right actions.

Definition. If $f : M_1 \to M_2$ is a $\mathbb{Z}G$-map, its *dual maps* $f^\star$ are $\mathbb{Z}G$-maps $\mathrm{Hom}_G(M_2, N) \to \mathrm{Hom}_G(M_1, N)$ for each $\mathbb{Z}G$-module N, given by composition on the right with f. If $f : N_1 \to N_2$, its *induced maps* $f_\star$ are $\mathrm{Hom}_G(M, N_1) \to \mathrm{Hom}_G(M, N_2)$ given by composition on the left with f. These are maps of abelian groups.

We will now present a prototypical example.

Example. Let $G = \langle t \rangle$ be an infinite cyclic group. Consider the graph whose vertices are v_i for $i \in \mathbb{Z}$, where v_i is joined to v_{i+1} and v_{i-1}. Let V be its set of vertices, and E be its set of edges. G acts by translations on this graph, where t maps v_i to v_{i+1}. The formal sums $\mathbb{Z}V$ and $\mathbb{Z}E$ can be regarded as $\mathbb{Z}G$-modules. They are free: $\mathbb{Z}V = \mathbb{Z}G\{v_0\}$, and $\mathbb{Z}E = \mathbb{Z}G\{e\}$ where e is the edge connecting v_0 and v_1. The boundary map is a $\mathbb{Z}G$-map $d : \mathbb{Z}E \to \mathbb{Z}V$ given by $e \mapsto v_1 - v_0$. There is also a $\mathbb{Z}G$-map $\mathbb{Z}V \to \mathbb{Z}$ given by $v_0 \mapsto 1$; this corresponds to the augmentation map.

Definition. A *chain complex* of $\mathbb{Z}G$-modules is a sequence

$$M_s \xrightarrow{d_s} M_{s-1} \xrightarrow{d_{s-1}} M_{s-2} \longrightarrow \cdots \xrightarrow{d_{t+1}} M_t$$

such that for every $t < n < s$, we have $d_n d_{n+1} = 0$, and so $\mathrm{im}\, d_{n+1} \subseteq \ker d_n$. We will refer to the entire sequence as $M_\bullet = (M_n, d_n)_{t \leq n \leq s}$.

We say that $M_\bullet$ is *exact* at M_n if $\mathrm{im}\, d_{n+1} = \ker d_n$, and we say it is *exact* if it is exact at all M_n for $t < n < s$. The *homology* of this chain complex is

$$H_s(M_\bullet) = \ker d_s; \quad H_n(M_\bullet) = {}^{\ker d_n}\!/_{\mathrm{im}\, d_{n+1}}; \quad H_t(M_\bullet) = \mathrm{coker}\, d_{t-1} = {}^{M_t}\!/_{\mathrm{im}\, d_{t+1}}$$

A *short exact sequence* is an exact chain complex of the form

$$0 \longrightarrow M_1 \xrightarrow{\alpha} M_2 \xrightarrow{\beta} M_3 \longrightarrow 0$$

That is, α is injective, β is surjective, and $\mathrm{im}\, \alpha = \ker \beta$.

Example. In our example above, we have the short exact sequence

$$0 \longrightarrow \mathbb{Z}E \longrightarrow \mathbb{Z}V \longrightarrow \mathbb{Z} \longrightarrow 0$$

This corresponds to a short exact sequence

$$0 \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

where $G = \langle t \rangle$ is an infinite cyclic group, and the map $\mathbb{Z}G \to \mathbb{Z}G$ is given by multiplication on the right by $t - 1$.

Definition. A $\mathbb{Z}G$-module P is *projective* if, for every surjective $\mathbb{Z}G$-map $\alpha : M_1 \to M_2$ and every $\mathbb{Z}G$-map $\beta : P \to M_2$, there is a map $\overline{\beta} : P \to M_1$ such that $\alpha \circ \overline{\beta} = \beta$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\overline{\beta}"'] \arrow[d, "\beta"] & \\ M_1 \arrow[r, "\alpha"'] & M_2 \arrow[r] & 0 \end{tikzcd}$$

Given any short exact sequence

$$0 \longrightarrow N \xrightarrow{f} M_1 \xrightarrow{\alpha} M_2 \longrightarrow 0$$

we can consider

$$0 \longrightarrow \mathrm{Hom}_G(P, N) \xrightarrow{f_*} \mathrm{Hom}_G(P, M_1) \xrightarrow{\alpha_*} \mathrm{Hom}_G(P, M_2) \longrightarrow 0$$

We could have defined projectivity by saying that this new sequence is exact. Note that this sequence is always a chain complex regardless if P is projective, and we always have exactness except possibly at $\mathrm{Hom}_G(P, M_2)$.

Lemma. Free modules are projective.

Proof. Let $\alpha : M_1 \to M_2$ be a surjective $\mathbb{Z}G$-map, and let $\beta : \mathbb{Z}G\{X\} \to M_2$. Then for each generator $x \in X$, there exists some $m_x \in M_1$ such that $\alpha(m_x) = \beta(x)$. We then define $\overline{\beta} : \mathbb{Z}G\{X\} \to M_1$ by mapping

$$\sum r_x x \mapsto \sum r_x m_x$$

which satisfies the required equation $\alpha\overline{\beta} = \beta$. $\square$

Definition. A *projective (free) resolution* of the trivial module $\mathbb{Z}$ is an exact sequence

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0$$

where the P_i are projective (respectively free). This is a chain complex.

Example. Let $G = \langle t \rangle$ be an infinite cyclic group. Then we have a finite free resolution of $\mathbb{Z}$ given by the exact sequence

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{\cdot (t-1)} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where ε is the augmentation map.

Example. Let $G = \langle t \rangle$ be a cyclic group of order n. Then we have a resolution

$$\cdots \longrightarrow \mathbb{Z}G \xrightarrow{\beta} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}G \xrightarrow{\beta} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where

$$\alpha(x) = x(t-1); \quad \beta(x) = x(1 + t + \cdots + t^{n-1})$$

From algebraic topology, if we have a connected simplicial complex X with fundamental group $\pi_1(X) = G$, such that the universal cover $\widetilde{X}$ is contractible, we obtain a free resolution of $\mathbb{Z}$ given by the universal cover. In this way, the simplicial complex X contains a lot of information about its fundamental group; this is what we aim to replicate algebraically.

For calculation purposes, we are interested in 'small' resolutions, for instance where the free modules have small rank. However, for theory development, we often want general constructions, and resolutions given by generic theory tend to be large.

Definition. G is of *type FP_n* if $\mathbb{Z}$ has a projective resolution

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0$$

which may be infinite, but where $P_n, P_{n-1}, \dots, P_0$ are finitely generated as $\mathbb{Z}G$-modules.
We say G is of *type FP_∞* if $\mathbb{Z}$ has a projective resolution where all of the P_i are finitely generated as $\mathbb{Z}G$-modules. Finally, G is of *type FP* if $\mathbb{Z}$ has a projective resolution where all of the P_i are finitely generated as $\mathbb{Z}G$-modules, and the resolution is of finite length, so $P_s = 0$ for sufficiently large s.

Example. (i) Let $G = \langle t \rangle$ be the infinite cyclic group. Then G is of type FP.

(ii) Let $G = \langle t \rangle$ be a finite cyclic group. Then G is of type FP_∞; we will show later that it is not of type FP.

These can be regarded as finiteness conditions on the group G. The topological version of FP_n would be that a simplicial complex X with fundamental group G has a finite n-skeleton.

1.2 ???

Consider a partial projective resolution

$$P_s \longrightarrow P_{s-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

Then we can set P_{s+1} to be the free module $\mathbb{Z}G\{X_{s+1}\}$ where X_{s+1} is the kernel of d_s. We can then set d_{s+1} to be

$$\underbrace{\sum r_x x}_{\in P_{s+1}} \mapsto \underbrace{\sum r_x x}_{\in P_s}$$

where the left-hand side is a formal sum, and the right-hand sum takes place in P_s. We thus obtain a longer partial projective resolution

$$P_{s+1} \xrightarrow{d_{s+1}} P_s \longrightarrow P_{s-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

since exactness holds at P_s by construction. We could alternatively take X_{s+1} to be a $\mathbb{Z}G$-generating set of $\ker d_s$; this would have the effect of reducing the size of P_{s+1}, which is most useful in direct calculation if $\ker d_s$ is finitely generated. Continuing in this way, we obtain a resolution of $\mathbb{Z}$.

Definition. The *standard* or *bar* resolution of $\mathbb{Z}$ is constructed as follows. Let $G^{(n)}$ be the set of formal symbols

$$G^{(n)} = \{[g_1| \dots |g_n] \mid g_1, \dots, g_n \in G\}$$

where $G^{(0)}$ is the set containing only the empty symbol $[]$. Let $F_n = \mathbb{Z}G\{G^{(n)}\}$ be the corresponding free modules. We define the boundary maps $d_n : F_n \to F_{n-1}$ by

$$\begin{aligned} d_n([g_1| \dots |g_n]) = {} & g_1[g_2| \dots |g_n] \\ & - [g_1g_2|g_3| \dots |g_n] \\ & + [g_1|g_2g_3| \dots |g_n] - \dots \\ & + (-1)^{n-1}[g_1| \dots |g_{n-1}g_n] \\ & + (-1)^n[g_1| \dots |g_{n-1}] \end{aligned}$$

One can verify explicitly that these are chain maps as required, giving a free resolution

$$\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z}$$

Remark. The bar resolution corresponds to the standard resolution in algebraic topology. Consider the free abelian group $\mathbb{Z}G^{n+1}$ generated by the $(n+1)$-tuples with elements in G. Then G acts on G^{n+1} diagonally:

$$g(g_0, \dots, g_n) = (gg_0, \dots, gg_n)$$

Thus $\mathbb{Z}G^{n+1}$ is a free $\mathbb{Z}G$-module on the basis of $(n+1)$-tuples with first element 1. The symbol $[g_1| \dots |g_n]$ corresponds to the $(n+1)$-tuple

$$(1, g_1, g_1g_2, \dots, g_1 \dots g_n)$$

Removing the first entry gives

$$g_1(1, g_2, g_2g_3, \dots, g_2 \dots g_n)$$

and removing the second entry gives

$$(1, g_1g_2, \dots, g_1 \dots g_n)$$

Lemma. The bar resolution is exact.

Proof. We will just consider the d_n as maps of abelian groups. F_n has basis $G \times G^{(n)}$ as a free abelian group.

$$G \times G^{(n)} = \{g_0[g_1| \dots |g_n] \mid g_0, \dots, g_n \in G\}$$

We define $\mathbb{Z}$-maps $s_n : F_n \to F_{n+1}$ such that

$$\mathrm{id}_{F_n} = d_{n+1}s_n + s_{n-1}d_n$$

by

$$s_n(g_0[g_1| \dots |g_n]) = [g_0|g_1| \dots |g_n]$$

This is not a $\mathbb{Z}G$-map. One can check that the required equation holds. If $x \in \ker d_n$, then

$$x = \operatorname{id} x = d_{n+1}s_n(x) + s_{n-1}d_n(x) = d_{n+1}s_n(x) \in \operatorname{im} d_{n+1}$$

□

Corollary. Any finite group is of type FP_∞.

Proof. The bar resolution gives a suitable resolution. □

1.3 Cohomology

Definition. Consider a projective resolution

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

of $\mathbb{Z}$ by $\mathbb{Z}G$-modules. Let M be a (left) $\mathbb{Z}G$-module. Applying $\operatorname{Hom}_G(-, M)$, we obtain a sequence

$$\cdots \longleftarrow \operatorname{Hom}_G(P_{n+1}, M) \longleftarrow \operatorname{Hom}_G(P_n, M) \longleftarrow \cdots \longleftarrow \operatorname{Hom}_G(P_1, M) \xleftarrow{d^1} \operatorname{Hom}_G(P_0, M)$$

where $d^n = d_n^\star$. Then the *nth cohomology group* $H^n(G, M)$ with coefficients in M is

$$H^n(G, M) = \ker d^{n+1} \operatorname{im} d^n; \quad H^0(G, M) = \ker d^1$$

Remark. We have removed the $\mathbb{Z}$ term in the $\operatorname{Hom}_G(-, M)$ sequence. These cohomology groups are the homology groups of a chain complex $C_n = \operatorname{Hom}_G(P_{-n}, M)$ for $n \leq 0$. We will show that these cohomology groups are independent of the choice of projective resolution.

Example. Let $G = \langle t \rangle$ be an infinite cyclic group. We have a projective resolution

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{\cdot (t-1)} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

For $\varphi \in \operatorname{Hom}_G(\mathbb{Z}G, M)$ and $x \in \mathbb{Z}G$,

$$d^1(\varphi)(x) = \varphi(d_1(x)) = \varphi(x(t-1))$$

Recall that we have an isomorphism $i : \operatorname{Hom}_G(\mathbb{Z}G, M) \cong M$ by $\theta \mapsto \theta(1)$. In particular,

$$d^1(\varphi) \mapsto d^1(\varphi)(1) = \varphi(t-1) = (t-1)\varphi(1) = (t-1)i(\varphi)$$

We thus obtain

$$0 \longleftarrow M \xleftarrow{\alpha} M$$

where α is multiplication on the left by $t - 1$. Therefore, the cohomology groups are

$$H^0(G, M) = \{m \in M \mid tm = m\} = M^G; \quad H^1(G, M) = {}^{M}\!\big/_{(t-1)M} = M_G; \quad H^n(G, M) = 0 \text{ for } n \neq 0, 1$$

Note that the group of invariants M^G is the largest submodule with trivial G-action, and the group of coinvariants M_G is the largest quotient module with trivial G-action.

Remark. It is generally true that $H^0(G, M) = M^G$, but in general $H^1(G, M) = M_G$ does not hold. In general, M_G is the 0th homology group, which will be discussed later. Note that for any group of type FP, the cohomology groups vanish for all but finitely many indices n.

Definition. G is of *cohomological dimension m* over $\mathbb{Z}$ if there exists some $\mathbb{Z}G$-module M with $H^m(G, M) \neq 0$ but $H^n(G, M_1) = 0$ for all $n > m$ and all $\mathbb{Z}G$-modules M_1.

Remark. For all G, we have $H^0(G, \mathbb{Z}) = \mathbb{Z} \neq 0$ so all groups have dimension at least zero.

Example. Infinite cyclic groups have cohomological dimension 1 over $\mathbb{Z}$. One can show that if G is a free group of finite rank, then it is also of cohomological dimension 1 over $\mathbb{Z}$. Stallings showed in 1968 that the converse is true: a finitely generated group of cohomological dimension 1 is free. Swan strengthened this in 1969 by removing the assumption of finite generation.

We now consider the bar resolution in our definition of cohomology. Note that

$$\mathrm{Hom}_G(\mathbb{Z}G\{G^{(n)}\}, M) \cong C^n(G, M)$$

where $C^n(G, M)$ is the set of functions $G^{(n)} \to M$, since a $\mathbb{Z}G$-map is determined by its action on a basis. Moreover, $C^n(G, M)$ corresponds to the set of functions $G^n \to M$. For $n = 0$, note that $C^0(G, M)$ is the set of functions $G^0 \to M$ which bijects with M.

Definition. The abelian group of *n-cochains* of G with coefficients in M is $C^n(G, M)$. The *nth coboundary map* $d^n : C^{n-1}(G, M) \to C^n(G, M)$ is dual to the d_n from the bar resolution:

$$\begin{aligned} d^n(\varphi)(g_1, \dots, g_n) &= g_1\varphi(g_2, \dots, g_n) \\ &\quad - \varphi(g_1g_2, g_3, \dots, g_n) \\ &\quad + \varphi(g_1, g_2g_3, \dots, g_n) - \cdots \\ &\quad + (-1)^{n-1}\varphi(g_1, g_2, \dots, g_{n-1}g_n) \\ &\quad + (-1)^n\varphi(g_1, g_2, \dots, g_{n-1}) \end{aligned}$$

The group of *n-cocycles* is $Z^n(G, M) = \ker d^{n+1} \leq C^n(G, M)$. The group of *$n$-coboundaries* is $B^n(G, M) = \operatorname{im} d^n \leq C^n(G, M)$. Thus the nth cohomology group is

$$H^n(G, M) = Z^n(G, M) \big/ B^n(G, M)$$

Corollary. $H^0(G, M) = M^G$ for all G.

Definition. A *derivation* of G with coefficients in M is a function $\varphi : G \to M$ such that $\varphi(gh) = g\varphi(h) + \varphi(g)$.

Note that $Z^1(G, M)$ is exactly the set of derivations of G with coefficients in M, so a derivation is precisely a 1-cocycle.

Definition. An *inner derivation* of G with coefficients in M is a function $\varphi : G \to M$ of the form $\varphi(g) = gm - m$ for a fixed $m \in M$.

Such maps are derivations.

Corollary. $H^1(G,M)$ is the group of derivations modulo the inner derivations. In particular, if M is a trivial $\mathbb{Z}G$-module, then

$$H^1(G,M) = \{\text{group homomorphisms } G \to M\}$$

treating M as an abelian group under addition.

1.4 Independence of cohomology groups

We now prove that cohomology groups are independent of the choice of resolution.

Definition. Let $(A_n, \alpha_n), (B_n, \beta_n)$ be chain complexes of $\mathbb{Z}G$-modules. A *chain map* (f_n) is a sequence of $\mathbb{Z}G$-maps $f_n : A_n \to B_n$ such that the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "\alpha_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d, "f_{n-1}"] & A_{n-2} \arrow[r] \arrow[d, "f_{n-2}"] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "\beta_n"'] & B_{n-1} \arrow[r, "\beta_{n-1}"'] & B_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

Lemma. A chain map (f_n) as above induces a map on homology groups

$$f_\star : H_n(A_\bullet) \to H_n(B_\bullet)$$

Proof. Let $x \in \ker \alpha_n$, and define $f_\star([x]) = [f_n(x)]$, where square brackets denote the quotient maps to the relevant homology classes. Observe that $f_n(x) \in \ker \beta_n$, since $\beta_n f_n(x) = f_{n-1}\alpha_n(x) = 0$. Further, if $x' = x + \alpha_{n+1}(y)$ for some y, we obtain

$$f_n(x') = f_n(x) + f_n\alpha_{n+1}(y) = f_n(x) + \beta_{n+1}f_{n+1}(y) \in f_n(x) + \operatorname{im} b_{n+1}$$

Therefore, this map is well-defined. One can check that this is a map of abelian groups, as required. □

Theorem. The definition of $H^n(G,M)$ does not depend on the choice of resolution.

Proof. Take projective resolutions (P_n, d_n) and (P_n', d_n') of $\mathbb{Z}$ by projective $\mathbb{Z}G$-modules. We will produce $\mathbb{Z}G$-maps $f_n : P_n \to P_n'$ and $g_n : P_n' \to P_n$ satisfying

$$f_{n-1}d_n = d_n' f_n; \quad g_{n-1}d_n' = d_n g_n$$

as well as maps $s_n : P_n \to P_{n+1}$ and $s_n' : P_n' \to P_{n+1}'$ satisfying

$$d_{n+1}s_n + s_{n-1}d_n = g_n f_n - \mathrm{id}; \quad d_{n+1}'s_n' + s_{n-1}'d_n' = f_n g_n - \mathrm{id}$$

Thus, the f_n and g_n form chain maps, and the s_n and s'_n form *chain homotopies*. The chain maps $(f_n), (g_n)$ give rise to chain maps

$$\mathrm{Hom}_G(P'_\bullet, M) \to \mathrm{Hom}_G(P_\bullet, M); \quad \mathrm{Hom}_G(P_\bullet, M) \to \mathrm{Hom}_G(P'_\bullet, M)$$

giving maps between the respective homology groups by the previous lemma. We now observe that if $\varphi \in \ker d^{n+1} \in \mathrm{Hom}(P, M)$, we have

$$\begin{aligned} f_n^\star g_n^\star(\varphi)(x) &= \varphi(g_n f_n(x)) \\ &= \varphi(x) + \varphi(d_{n+1}s_n(x)) + \varphi(s_{n-1}d_n(x)) \\ &= \varphi(x) + s_n^\star d^{n+1}\varphi(x) + d^n s_{n-1}^\star(\varphi)(x) \\ &= \varphi(x) + 0 + d^n s_{n-1}^\star(\varphi)(x) \end{aligned}$$

Thus $f_n^\star g_n^\star(\varphi) = \varphi + d^n s_{n-1}^\star(\varphi)$, and so $f_n^\star g_n^\star$ induces the identity map on ${}^{\ker d^{n+1}}\!/_{\mathrm{im}\, d^n}$. The same holds for $g_n^\star f_n^\star$, and so $f_n^\star, g_n^\star$ define isomorphisms of homology groups as desired.

It remains to construct the maps f_n, g_n, s_n, s'_n. At the end of the resolutions, we set $f_{-1} : \mathbb{Z} \to \mathbb{Z}$ and $f_{-2} : 0 \to 0$ to be the identity maps. Suppose that we have already defined f_{n-1} and f_n; we will define f_{n+1}. We have $f_n d_{n+1} : P_{n+1} \to P'_n$ and $d'_n \circ (f_n d_{n+1}) = f_{n-1} d_n d_{n+1} = 0$. Hence, the map $f_n d_{n+1}$ has image inside $\ker d'_n$. We then define f_{n+1} to complete the following diagram, which exists by projectivity.

$$\begin{tikzcd}
 & P_{n+1} \arrow[r, "d_{n+1}"] \arrow[dl, dashed, "f_{n+1}"'] \arrow[d, "f_n d_{n+1}"] & P_n \arrow[r, "d_n"] \arrow[dd, "f_n"] & P_{n-1} \arrow[dd, "f_{n-1}"] \\
P'_{n+1} \arrow[r, two heads, "d'_{n+1}"'] & \ker d'_n \arrow[r, tail] & P'_n \arrow[r, "d'_n"'] & P'_{n-1}
\end{tikzcd}$$

We can define g_{n+1} in the same way. Now set $h_n = g_n f_n - \mathrm{id} : P_n \to P_n$; this gives a chain map $P_\bullet \to P_\bullet$. Set $s_{-1} : \mathbb{Z} \to P_0$ to be the zero map. Note that $d_0 h_0 = h_{-1} d_0 = 0$, and so $\mathrm{im}\, h_0 \subseteq \ker d_0$. We now use projectivity to define

$$\begin{tikzcd}
 & P_0 \arrow[r] \arrow[dl, dashed, "s_0"'] \arrow[d, "h_0"'] \arrow[dr, "h_0"] & \mathbb{Z} \arrow[dr, "0"] & \\
P_1 \arrow[r, two heads, "d_1"'] & \ker d_0 \arrow[r, tail] & P_0 \arrow[r, "d_0"'] & \mathbb{Z}
\end{tikzcd}$$

Suppose that s_{n-1} and s_{n-2} are already defined. Consider $t_n = h_n - s_{n-1}d_n : P_n \to P_n$. We have

$$d_n t_n = d_n h_n - d_n s_{n-1} d_n = h_{n-1} d_n - (h_{n-1} - s_{n-2} d_{n-1}) d_n = s_{n-2} d_{n-1} d_n = 0$$

Thus $\mathrm{im}\, t_n \subseteq \ker d_n$.

$$\begin{tikzcd}
 & P_n \arrow[r, "d_n"] \arrow[dl, dashed, "s_n"'] \arrow[d, "t_n"'] \arrow[dr, "h_n"] & P_{n-1} \arrow[d, dashed, "s_{n-1}"] \arrow[dr, "h_{n-1}"] & \\
P_{n+1} \arrow[r, two heads] & \ker d_n \arrow[r, tail] & P_n \arrow[r, "d_n"'] & P_{n-1}
\end{tikzcd}$$

We define the s'_n similarly. $\square$

Remark. For any left $\mathbb{Z}G$-module N, we can take a resolution of N by projective or free $\mathbb{Z}G$-modules.

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow N \longrightarrow 0$$

Repeating the constructions outlined in this section, applying $\mathrm{Hom}_G(-, M)$ gives homology groups called $\mathrm{Ext}^n_{\mathbb{Z}G}(N, M)$. Thus

$$H^n(G, M) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, M)$$

2 Low degree cohomology and group extensions

2.1 Degree 1

Recall that $H^0(G, M)$, the group M^G of invariants of M under G. A derivation is a 1-cocycle, or equivalently a map $\varphi : G \to M$ such that $\varphi(g_1g_2) = g_1\varphi(g_2) + \varphi(g_1)$, and an inner derivation is a map of the form $\varphi(g) = gm - m$. We present two interpretations of (inner) derivations.

First interpretation. Consider possible $\mathbb{Z}G$-actions on the abelian group $M \oplus \mathbb{Z}$ of the form $g(m, n) = (gm + n\varphi(g), n)$. Then

$$g_1(g_2(m, n)) = g_1(g_2m + n\varphi(g_2), n) = (g_1g_2m + ng_1\varphi(g_2) + n\varphi(g_1), n)$$

and

$$(g_1g_2)(m, n) = (g_1g_2m + n\varphi(g_1g_2), n)$$

For these to coincide, we must require $\varphi(g_1g_2) = g_1\varphi(g_2) + \varphi(g_1)$, which is to say that φ is a derivation. In particular, if M is a free $\mathbb{Z}$-module of finite rank, then we obtain a map

$$g \mapsto \begin{pmatrix} \theta_1(g) & \varphi(g) \\ 0 & 1 \end{pmatrix}$$

where $\theta_1(g)$ is a matrix corresponding to the action of g on M. This is a group homomorphism only if φ is a derivation. One can check that φ is an inner derivation if $(-m, 1)$ generates a $\mathbb{Z}G$-submodule of M which is the trivial module.

Second interpretation. We first make the following definition.

Definition. Let G be a group and M be a left $\mathbb{Z}G$-module. We construct the *semidirect product* $M \rtimes G$ by defining a group operation on the set $M \times G$ as follows.

$$(m_1, g_1) * (m_2, g_2) = (m_1 + g_1m_2, g_1g_2)$$

Then $M \cong \{(m, 1) \mid m \in M\}$ is a normal subgroup of $M \rtimes G$. Also, $G \cong \{(0, g) \mid g \in G\}$, and conjugation by $\{(0, g) \mid g \in G\}$ corresponds to the G-action on the module M. Further,

$$M \rtimes G \big/ \{(m, 1) \mid m \in M\} \cong G$$

There is a group homomorphism $s : G \to M \rtimes G$ given by $g \mapsto (0, g)$, such that $\pi_2 \circ s = \mathrm{id}$ where π_2 is the second projection. Such a map s is called a *splitting*. Given another splitting $s_1 : G \to M \rtimes G$ such that $\pi_2 \circ s_1 = \mathrm{id}$, we define $\psi_{s_1} : G \to M$ by

$$s_1(g) = (\psi_{s_1}(g), g) \in M \rtimes G$$

Then ψ_{s_1} is a 1-cocycle. Given two splittings s_1, s_2, the difference $\psi_{s_1} - \psi_{s_2}$ is a coboundary precisely when there exists m such that $(m, 1)s_1(g)(m, 1)^{-1} = s_2(g)$. Conversely, given a 1-cocycle $\varphi \in Z^1(G, M)$, there is a splitting $s_1 : G \to M \rtimes G$ such that $\varphi = \psi_{s_1}$.

Theorem. $H^1(G,M)$ bijects with the M-conjugacy classes of splittings.

2.2 Degree 2

Definition. Let G be a group and M be a $\mathbb{Z}G$-module. An *extension* of G by M is a group E with an exact sequence of group homomorphisms

$$0 \longrightarrow M \longrightarrow E \longrightarrow G \longrightarrow 1$$

M embeds into E, so its image (also called M) is an abelian normal subgroup of E. This is acted on by conjugation by E, and so we obtain an induced action of ${}^{E}\!/\!{}_{M} \cong G$, which must match the given G-action on M.

Example. The semidirect product $M \rtimes G$ is an extension of G by M.

$$0 \longrightarrow M \longrightarrow M \rtimes G \longrightarrow G \longrightarrow 1$$

In this case, the extension is called a *split extension*, since there is a splitting.

Definition. Two extensions are *equivalent* if there is a commutative diagram of homomorphisms

$$\begin{tikzcd} & & E \arrow[dd, dashed] \arrow[dr] & & \\ 0 \arrow[r] & M \arrow[ur] \arrow[dr] & & G \arrow[r] & 1 \\ & & E' \arrow[ur] & & \end{tikzcd}$$

If E, E' are equivalent extensions, then E and E' are isomorphic as groups. The converse is false.

Definition. A *central* extension is an extension where the given $\mathbb{Z}G$-module is a trivial module (that is, it has trivial G-action).

Proposition. Let E be an extension of G by M. If there is a splitting homomorphism $s_1 : G \to E$, then the extension is equivalent to

$$0 \longrightarrow M \longrightarrow M \rtimes G \longrightarrow G \longrightarrow 1$$

and thus $E \cong M \rtimes G$.

Theorem. Let G be a group and let M be a $\mathbb{Z}G$-module. Then there is a bijection from $H^2(G,M)$ to the set of equivalence classes of extensions of G by M.

Proof. Given an extension

$$0 \longrightarrow M \longrightarrow E \longrightarrow G \longrightarrow 1$$

there is a set-theoretic section $s : G \to E$ such that

$$\begin{tikzcd} G \arrow[r, "s"] \arrow[dr, "\mathrm{id}"'] & E \arrow[d, "\pi"] \\ & G \end{tikzcd}$$

commutes. Note that s need not be a group homomorphism. Without loss of generality, we can suppose $s(1) = 1$. We define a map

$$\varphi(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1}$$

which measures the failure of s to be a group homomorphism. Then $\pi(\varphi(g_1, g_2)) = 1$, and so $\varphi(g_1, g_2) \in M$. Thus $\varphi : G^2 \to M$ is a 2-cochain, and we can show it is a 2-cocycle. We have

$$\begin{aligned} s(g_1)s(g_2)s(g_3) &= \varphi(g_1, g_2)s(g_1g_2)s(g_3) \\ &= \varphi(g_1, g_2)\varphi(g_1g_2, g_3)s(g_1g_2g_3) \end{aligned}$$

and similarly,

$$\begin{aligned} s(g_1)s(g_2)s(g_3) &= s(g_1)\varphi(g_2, g_3)s(g_2g_3) \\ &= s(g_1)\varphi(g_2, g_3)s(g_1)^{-1}s(g_1)s(g_2g_3) \\ &= s(g_1)\varphi(g_2, g_3)s(g_1)^{-1}\varphi(g_1, g_2g_3)s(g_1g_2g_3) \end{aligned}$$

We therefore obtain

$$\begin{aligned} \varphi(g_1, g_2)\varphi(g_1g_2, g_3)s(g_1g_2g_3) &= s(g_1)\varphi(g_2, g_3)s(g_1)^{-1}\varphi(g_1, g_2g_3)s(g_1g_2g_3) \\ \varphi(g_1, g_2)\varphi(g_1g_2, g_3) &= s(g_1)\varphi(g_2, g_3)s(g_1)^{-1}\varphi(g_1, g_2g_3) \end{aligned}$$

Converting into additive notation,

$$\varphi(g_1, g_2) + \varphi(g_1g_2, g_3) = g_1\varphi(g_2, g_3) + \varphi(g_1, g_2g_3)$$

and so

$$(d^3\varphi)(g_1, g_2, g_3) = 0$$

Hence φ is a 2-cocycle as claimed. Note that φ is a *normalised* cocycle: it satisfies $\varphi(1, g) = \varphi(g, 1) = 0$. We have therefore proven that an extension of G by M, with a choice of set-theoretic section $s : G \to E$, yields a normalised 2-cocycle $\varphi \in Z^2(G, M)$.

Now take another choice of section s' with $s'(1) = 1$. We show that the normalised cocycles φ, φ' differ by a coboundary, and so we have a map defined from equivalence classes of extensions to $H^2(G, M)$. We have $\pi(s(g)s'(g)^{-1}) = 1$, so $s(g)s'(g)^{-1} \in \ker \pi = M$. Let $\psi(g)$ denote $s(g)s'(g)^{-1}$. Thus $\psi : G \to M$. We have

$$\begin{aligned} s'(g_1)s'(g_2) &= \psi(g_1)s(g_1)\psi(g_2)s(g_2) \\ &= \psi(g_1)s(g_1)\psi(g_2)s(g_1)^{-1}s(g_1)s(g_2) \\ &= \psi(g_1)s(g_1)\psi(g_2)s(g_1)^{-1}\varphi(g_1, g_2)s(g_2) \\ &= \psi(g_1)s(g_1)\psi(g_2)s(g_1)^{-1}\varphi(g_1, g_2)\psi(g_1g_2)^{-1}s'(g_1g_2) \end{aligned}$$

Switching to additive notation,

$$\begin{aligned}\varphi'(g_1,g_2) &= \psi(g_1) + g_1\psi(g_2) + \varphi(g_1,g_2) - \psi(g_1g_2)\\ &= \varphi(g_1,g_2) + (d^2\psi)(g_1,g_2)\end{aligned}$$

Thus φ and φ' differ by a coboundary, and so we have a well-defined map from extensions of G by M to $H^2(G,M)$.

To complete the proof, we must check that equivalent extensions give rise to the same cohomology class, and that there is an inverse map from cohomology classes to equivalence classes of extensions. To produce the inverse, we use the following lemma.

Lemma. Let $\varphi \in Z^2(G,M)$. Then there is a cochain $\psi \in C^1(G,M)$ such that $\varphi + d^2\psi$ is a normalised cocycle. Hence, every cohomology class can be represented by a normalised cocycle.

Proof. Let $\psi(g) = -\varphi(1,g)$. Then

$$\begin{aligned}(\varphi + d^2\psi)(1,g) &= \varphi(1,g) - (\varphi(1,g) - \varphi(1,g) + \varphi(1,1))\\ &= \varphi(1,g) - \varphi(1,1)\end{aligned}$$

Similarly, we obtain

$$(\varphi + d^2\psi)(g,1) = \varphi(g,1) - g\varphi(1,1)$$

But we know that

$$d^3\varphi(1,1,g) = 0 = d^3\varphi(g,1,1)$$

since φ is a cocycle. Hence, one can check computationally that both equations above are zero. □

We now take a normalised cocycle φ representing a given cohomology class. We construct an extension

$$0 \longrightarrow M \longrightarrow E_\varphi \longrightarrow G \longrightarrow 1$$

by

$$(m_1,g_1) * (m_2,g_2) = (m_1 + g_1m_2 + \varphi(g_1,g_2), g_1,g_2)$$

For this to be a group operation, we use the fact that φ is normalised. This yields an extension

$$0 \longrightarrow M \longrightarrow E_\varphi \xrightarrow{\pi} G \longrightarrow 1$$

where π is the projection onto the second component. Note that if φ' is another normalised 2-cocycle representing the given cohomology class, then $\varphi - \varphi'$ is a coboundary, so we can define a map $E_\varphi \to E_{\varphi'}$ by

$$(m,g) \mapsto (m + \psi(g), g)$$

One can check that this induces an equivalence of extensions. These constructions are inverses. □

2.3 Central extensions

Example. Consider central extensions of $\mathbb{Z}^2$ by $\mathbb{Z}$. We already know of two such extensions. The first is

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2 \longrightarrow 0$$

$$m \longmapsto (m, 0, 0)$$

$$(m, r, s) \longmapsto (r, s)$$

Let H denote the *Heisenberg group*

$$H = \left\{ \begin{pmatrix} 1 & r & m \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \middle| r, s, m \in mathbbZ \right\}$$

Then we have the extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow H \longrightarrow \mathbb{Z}^2 \longrightarrow 0$$

$$m \longmapsto \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & r & m \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \longmapsto (r, s)$$

Writing multiplicatively, let $T \cong \mathbb{Z}^2$ be generated by a and b. We have the following free resolution of the trivial $\mathbb{Z}T$-module $\mathbb{Z}$.

$$0 \longrightarrow \mathbb{Z}T \xrightarrow{\beta} \mathbb{Z}T^2 \xrightarrow{\alpha} \mathbb{Z}T \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where

$$\beta(z) = (z(1-b), z(a-1))$$
$$\alpha(x, y) = x(a-1) + y(b-1)$$

and ε is the augmentation map. Apply $\mathrm{Hom}_T(-, \mathbb{Z})$ to obtain the chain complex

$$0 \longleftarrow \mathrm{Hom}_T(\mathbb{Z}T, \mathbb{Z}) \xleftarrow{\beta^\star} \mathrm{Hom}_T(\mathbb{Z}T^2, \mathbb{Z}) \xleftarrow{\alpha^\star} \mathrm{Hom}_T(\mathbb{Z}T, \mathbb{Z})$$

We claim that $\alpha^\star$ and $\beta^\star$ are both zero maps, and so

$$H^2(T, \mathbb{Z}) = \mathrm{Hom}_T(\mathbb{Z}T, \mathbb{Z}) \cong \mathbb{Z}$$

and the generator is represented by the augmentation map $\varepsilon : \mathbb{Z}T \to \mathbb{Z}$.

Take a $\mathbb{Z}T$-map $f : \mathbb{Z}T^2 \to \mathbb{Z}$. Then

$$\begin{aligned}(\beta^\star f)(z) &= f(\beta)(z)\\ &= f(z(1-b), z(a-1))\\ &= f(z-zb,0) + f(0,za-z)\\ &= (1-b)f(z,0) + (a-1)f(0,z)\\ &= 0\end{aligned}$$

where the last line holds as T acts trivially. Similarly, $\alpha^\star = 0$.

Next, we interpret $H^2(T,\mathbb{Z})$ in terms of 2-cocycles arising from the bar resolution. We construct a chain map as follows.

$$\begin{tikzcd}
\mathbb{Z}T\{T^{(2)}\} \arrow[r, "d_2"] \arrow[d, "f_2"'] & \mathbb{Z}T\{T^{(1)}\} \arrow[r, "d_1"] \arrow[d, "f_1"'] & \mathbb{Z}T\{T^{(0)}\} \arrow[r, "\varepsilon"] \arrow[d, "\mathrm{id}"'] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal]\\
\mathbb{Z}T \arrow[r, "\beta"'] & \mathbb{Z}T^2 \arrow[r, "\alpha"'] & \mathbb{Z}T \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

To construct f_1 such that $\alpha f_1 = d_1$, we need to give images of the symbols $[a^r b^s]$ with $r,s \in \mathbb{Z}$. We must have

$$[a^r b^s] \mapsto (x_{r,s}, y_{r,s}) \in \mathbb{Z}T^2$$

where

$$\alpha(x_{r,s}, y_{r,s}) = d_1([a^r b^s]) = a^r b^s - 1 = (a^r-1)b^s + (b^s - 1)$$

We define

$$S(a,r) = \begin{cases} 1 + a + \cdots + a^{r-1} \\ \text{if } r > 0 \\ -a^{-1} - \cdots - a^r \\ textif r \leq 0 \end{cases}$$

Note that

$$S(a,r)(a-1) = a^r - 1$$

for any $r \in \mathbb{Z}$. Then

$$\begin{aligned}\alpha(S(a,r)b^s, S(b,s)) &= S(a,r)b^s(a-1) + S(b,s)(b-1)\\ &= d_1([a^r b^s])\end{aligned}$$

as required. So we may define

$$f_1([a^r b^s]) = (S(a,r)b^s, S(b,s))$$

To define f_2, we need to give images of the symbols $[a^r b^s | a^t b^u]$. For each such symbol, we find $z_{r,s,t,u} \in \mathbb{Z}T$ such that

$$f_1 d_2([a^r b^s | a^t b^u]) = \beta(z_{r,s,t,u})$$

We can explicitly calculate

$$\begin{aligned}f_1 d_2([a^r b^s | a^t b^u]) &= f_1(a^r b^s [a^t b^u] - [a^{r+t} b^{s+u}] - [a^r b^s])\\ &= (a^r b^s S(a,t) b^u - S(a,r+t)b^{s+u} + S(a,r)b^s, a^r b^s S(b,u) - S(b,s+u) + S(b,s))\end{aligned}$$

So defining

$$z_{r,s,t,u} = S(a,r)b^s S(b,u)$$

gives the required equation.

$$f_2([a^r b^s | a^t b^u]) = S(a,r)b^s S(b,u)$$

Now we find a cochain $\varphi : T^2 \to \mathbb{Z}$ representing the cohomology class $p \in \mathbb{Z} = \mathrm{Hom}_T(\mathbb{Z}T, \mathbb{Z}) = H^2(T, \mathbb{Z})$. Such a cochain is given by the composition

$$T^2 \xrightarrow{f_2} \mathbb{Z}T \xrightarrow{p\varepsilon} \mathbb{Z}$$

Since $\varepsilon(S(a,r)) = r$, we find

$$\varphi(a^r b^s, a^t b^u) = p\varepsilon(z_{r,s,t,u}) = pru$$

The group structure on $\mathbb{Z} \times T$ corresponding to this is

$$(m, a^r b^s) * (n, a^t b^u) = (m + n + pru, a^{r+t} b^{s+u})$$

This corresponds to the group of matrices

$$\left\{ \begin{pmatrix} 1 & pr & m \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix} \middle| \, r, s, m \in \mathbb{Z} \right\}$$

2.4 Generators and relations

Another approach to considering extensions, and in particular central extensions, is the use of partial resolutions arising from generators and relations. Given a group G, for any generating set X there is a canonical map $F \to G$ where F is the free group on X. Let R be the kernel of this map, and so we have a short exact sequence

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

This is a presentation for G, where the subgroup R can be thought of as the set of relations. Since it is a normal subgroup, F acts on it by conjugation. Often we take a set of generators of R as a normal subgroup of F.

Let $R_{\mathrm{ab}} = {}^{R}\!/\!_{R'}$ be the largest abelian quotient of R. We say that R' is the *derived subgroup* of R, and is given by the commutator subgroup $[R, R]$ of F. It inherits an action of F, but R acts trivially, so we have an induced action by $G = {}^{F}\!/\!_{R}$. Clearly R_{ab} is a $\mathbb{Z}$-module, and it is a $\mathbb{Z}G$-module. This is called the *relation module*. We have an extension

$$1 \longrightarrow R_{\mathrm{ab}} \longrightarrow {}^{F}\!/\!_{R'} \longrightarrow G \longrightarrow 1$$

To get a central extension, we instead consider

$$1 \longrightarrow {}^{R}\!/\!_{[R,F]} \longrightarrow {}^{F}\!/\!_{[R,F]} \longrightarrow G \longrightarrow 1$$

where $[R, F]$ is the commutator subgroup. There is not a largest or universal central extension, since we can always form the direct product with an abelian group, but this particular central extension above does have some good properties that we will now explore.

Theorem. Let

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

be a presentation of G. Let M be a left $\mathbb{Z}G$-module. Then there is an exact sequence

$$H^1(F,M) \longrightarrow \mathrm{Hom}_G(R_{\mathrm{ab}},M) \longrightarrow H^2(G,M) \longrightarrow 0$$

Thus, any equivalence class of extensions of G by M corresponding to a cohomology class in $H^2(G,M)$ arises from a $\mathbb{Z}G$-map $R_{\mathrm{ab}} \to M$.

Note that M is a $\mathbb{Z}F$-module via the map $F \to G$.

Corollary. In the above situation, if M is a trivial $\mathbb{Z}G$-module, then we have an exact sequence

$$\mathrm{Hom}(F,M) \longrightarrow \mathrm{Hom}_G({}^{R}\!/_{[R,F]},M) \longrightarrow H^2(G,M) \longrightarrow 0$$

Proof. M is a trivial $\mathbb{Z}F$-module, so $H^1(F,M) = \mathrm{Hom}(F,M)$, which is a set of group homomorphisms to an abelian group, and any such morphism factors uniquely through the abelianisation so this is equal to $\mathrm{Hom}(F_{\mathrm{ab}},M)$. Similarly, $\mathrm{Hom}_G(R_{\mathrm{ab}},M) = \mathrm{Hom}_G\left({}^{R}\!/_{[R,F]},M\right)$. □

2.5 Homology groups

There is also a connection with homology groups. Given a projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, we can apply the map $\mathbb{Z} \otimes_{\mathbb{Z}G} -$ and obtain homology groups. The homology groups do not depend on the choice of resolution, and are written $H_n(G,\mathbb{Z})$.

Definition. The *Schur multiplier* $M(G)$ of a group G is the second homology group $H_2(G,\mathbb{Z})$.

Theorem (universal coefficients theorem)**.** Let G be a group and M be a trivial $\mathbb{Z}G$-module. Then there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1(G_{\mathrm{ab}},M) \longrightarrow H^2(G,M) \longrightarrow \mathrm{Hom}(M(G),M) \longrightarrow 0$$

where $\mathrm{Ext}^1(G_{\mathrm{ab}},M)$ arises from applying $\mathrm{Hom}_{\mathbb{Z}}(-,M)$ to a projective resolution of the abelian group G_{ab}.

Corollary. Suppose that $G = G'$, and so $G_{\mathrm{ab}} = 1$. Then $H^2(G,M) \cong \mathrm{Hom}(M(G),M)$.

In some texts, the Schur multiplier is defined to be $H^2(G,\mathbb{C}^\times)$, where $\mathbb{C}^\times$ is the a trivial module written multiplicatively. This approach can be useful when considering projective representations $G \to PGL(\mathbb{C})$. Such a map lifts to give a linear representation of central extension of G.

Theorem (Hopf's formula)**.** Given a presentation

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

we have

$$M(G) \cong {}^{F' \cap R}\!\big/_{[R,F]}$$

Note that this is not necessarily all of ${}^{F}\!\big/_{[R,F]}$, and this shows that ${}^{F' \cap R}\!\big/_{[R,F]}$ is independent of the choice of presentation.