Forcing and the Continuum Hypothesis

Cambridge University Mathematical Tripos: Part III

21st May 2024

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1 Set theoretic preliminaries

1.1 Introduction to independence results

Independence results are found across mathematical disciplines.

(i) The *parallel postulate* is independent from the other four postulates of Euclidean geometry. It states that for any given point not on a line, there is a unique line passing through that point that does not intersect the given line. In the 19th century, it was shown that the other four postulates are satisfied by hyperbolic geometry, but this postulate is not satisfied. This shows that the other four axioms are insufficient to prove the parallel postulate.

(ii) Let φ be the statement in the language of fields describing the existence of a square root of 2. We know that $\mathbb{Q}$ is a field satisfying $\neg\varphi$, and $\mathbb{Q}[\sqrt{2}]$ satisfies φ. The fields $\mathbb{Q}$ and $\mathbb{Q}[\sqrt{2}]$ are models of the theory of fields, one of which satisfies φ, and one of which satisfies $\neg\varphi$. This shows that the theory of fields does not prove φ or $\neg\varphi$. A similar result holds for the statement φ that says that there are no roots of $x^4 = -1$.

(iii) Gödel's incompleteness theorem implies that there must always be an independence result in a sufficiently powerful consistent set theory.

We will show that there are other independence results in set theory that are not self-referential like the Gödel incompleteness theorems.

Theorem (Cantor). $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$.

The continuum hypothesis is that there are no sets of cardinality strictly between $|\mathbb{N}|$ and $|\mathcal{P}(N)| = |\mathbb{R}|$.

Definition. The *continuum hypothesis* CH states that if $X \subseteq \mathcal{P}(\mathbb{N})$ is infinite, then either $|X| = |\mathbb{N}|$ or $|X| = |\mathcal{P}(\mathbb{N})|$, or equivalently,

$$2^{\aleph_0} = \aleph_1$$

Progress was made on the continuum hypothesis in the 19th and 20th centuries.

(i) In 1883, Cantor showed that any closed subset of $\mathbb{R}$ satisfies CH.

(ii) In 1916, Alexandrov and Hausdorff showed that any Borel set of $\mathbb{R}$ satisfies CH.

(iii) In 1930, Suslin strengthened this result to analytic subsets of $\mathbb{R}$.

(iv) In 1938, Gödel showed that if ZF is consistent, then so is ZFC + CH.

(v) However, in 1963, Cohen showed that if ZF is consistent, then so is ZFC + ¬CH.

In this course, we will prove results (iv) and (v), thus establishing the independence of the continuum hypothesis from ZFC.

1.2 Systems of set theory

The language of set theory $\mathcal{L} = \mathcal{L}_\in$ is a first-order predicate logic with equality and membership as primitive relations. We assume the existence of infinitely many variables $v_1, v_2, \dots$ denoting sets.

We will only use the logical connectives $\vee$ and $\neg$ as well as the existential quantifier $\exists$. Conjunction, implication, and universal quantification can be defined in terms of disjunction, negation, and existential quantification.

We say that an occurrence of a variable x is *bound* in a formula φ if is in a quantifier $\exists x$ or lies in the scope of such a quantifier. An occurrence is called *free* if it is not bound. We write $\mathrm{FV}(\varphi)$ for the set of free variables of φ. We will write $\varphi(u_1, \dots, u_n)$ to emphasise the dependence of φ on its free variables $u_1, \dots, u_n$. By doing so, we will allow ourselves to freely change the names of the free variables, and assume that substituted variables are free. The syntax $\varphi(u_1, \dots, u_n)$ does not imply that u_i occurs freely, or even at all.

Some of the most common axioms of set theory are as follows.

(i) *Axiom of extensionality.*

$$\forall x.\, \forall y.\, (\forall z.\, (z \in x \leftrightarrow x \in y) \to x = y)$$

(ii) *Axiom of empty set.*

$$\exists x.\, \forall y \in x.\, y \neq y$$

(iii) *Axiom of pairing.*

$$\forall x.\, \forall y.\, \exists z.\, (x \in z \wedge y \in z)$$

(iv) *Axiom of union.*

$$\forall a.\, \exists x.\, \forall y.\, (y \in x \leftrightarrow \exists z \in a.\, y \in z)$$

(v) *Axiom of foundation.*

$$\forall x.\, (\exists y.\, y \in x \to \exists y \in x.\, \neg \exists z \in x.\, z \in y)$$

(vi) *Axiom scheme of separation.* For any formula φ,

$$\forall a.\, \exists x.\, \forall y.\, (y \in x \leftrightarrow (y \in a \wedge \varphi(y)))$$

(vii) *Axiom of infinity.*

$$\exists a.\, (\exists x.\, (x \in a) \wedge \forall x \in a.\, \exists y \in a.\, x \in y)$$

(viii) *Axiom of power set.*

$$\forall a.\, \exists x.\, \forall y.\, (y \in x \leftrightarrow \forall z.\, (z \in y \to z \in a))$$

(ix) *Axiom scheme of replacement.* For any formula φ,

$$\forall a.\, (\forall x \in a.\, \exists! y.\, \varphi(x, y) \to \exists b.\, \forall x \in a.\, \exists y \in b.\, \varphi(x, y))$$

(ix′) *Axiom scheme of collection.* For any formula φ,

$$\forall a.\, (\forall x \in a.\, \exists y.\, \varphi(x, y) \to \exists b.\, \forall x \in a.\, \exists y \in b.\, \varphi(x, y))$$

(x) *Axiom of choice.*

$$\forall X.\, \left(\varnothing \notin X \to \exists f : \left(X \to \bigcup X\right).\, \forall a \in X.\, f(a) \in a\right)$$

(x′) *Well-ordering principle.*

$$\forall a.\, \exists R.\, R \text{ is a well-ordering of } a$$

Some common set theories are as follows.

- *Zermelo set theory* Z consists of axioms (i) to (viii). Axioms (ix) and (ix′) are equivalent relative to Z.
- *Zermelo–Fraenkel set theory* ZF consists of axioms (i) to (ix). Axioms (x) and (x′) are equivalent relative to ZF.
- *Zermelo–Fraenkel set theory with choice* ZFC consists of axioms (i) to (x).
- *Zermelo–Fraenkel set theory without power set* ZF^- consists of axioms (i) to (vii), with the axiom of collection (ix′) instead of replacement (ix); it has been shown that (ix) is weaker than (ix′) in the presence of axioms (i) to (vii).
- *Zermelo–Fraenkel set theory with choice and without power set* ZFC^- consists of axioms (i) to (vii), with the axiom of collection (ix′) and the well-ordering principle (x′).

In this course, our main metatheory will be ZF, and we will be explicit about the use of choice.

We say that a class X is *definable* over M if there exists a formula φ and sets $a_1, \dots, a_n \in M$ such that for all $z \in M$, we have $z \in X$ if and only if $\varphi(z, a_1, \dots, a_n)$. A class is *proper* over M if it is not a set in M.

Under suitable hypotheses, there is a countable transitive model M of ZFC. In this case, $|\mathbb{R} \cap M|$ is countable, so there exists a real v that is not in M. Hence, v is a proper class over M. However, it is not definable, and we cannot 'talk about it' in the language of set theory. The only proper classes that affect our theory are the definable ones.

In this course, we will assume that all mentioned classes are definable. We can then use formulas of the form

$$\exists C.\, (C \text{ is a class} \wedge \forall x \in C.\, \varphi)$$

by defining it to mean that there is a formula θ giving a class C satisfying $\forall x \in C.\, \varphi$. For example, the universe class $\mathrm{V} = \{x \mid x = x\}$, the Russell class $R = \{x \mid x \notin x\}$, and the class of ordinals Ord are all definable. Any set is a definable class. Classes are heavily dependent on the underlying model: if $M = 2$ then $\mathrm{Ord} = 2 = M$, and if $M = 3 \cup \{1\}$ then $\mathrm{Ord} = 3 \neq M$.

Suppose that M is a set model of ZF; that is, M is a set. Let $\mathcal{D}$ be the collection of definable classes over M. Then one can show that $\mathcal{D}$ is a set in our metatheoretic universe V, and $(M, \mathcal{D})$ is a model of a second-order version of ZF, known as *Gödel–Bernays set theory*.

1.3 Adding defined functions

Often in set theory, we use symbols such as $0, 1, \subseteq, \cap, \wedge, \forall$; they do not exist in our language.

Definition. Suppose that $\mathcal{L} \subseteq \mathcal{L}'$ and T is a set of sentences in $\mathcal{L}$. We say that P is a *defined n-ary predicate* symbol over T if there is a formula φ in $\mathcal{L}$ such that

$$T \vdash \forall x_1, \dots, x_n.\, (P(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n))$$

Similarly, we say that f is a *defined n-ary function* symbol over T if there is a formula φ in $\mathcal{L}$

such that

$$f(x_1, \dots, x_n) = y \text{ if and only if } T \vdash \varphi(x_1, \dots, x_n, y)$$

and

$$T \vdash \forall x_1, \dots, x_n. \exists! y. \varphi(x_1, \dots, x_n, y)$$

We say that a set of sentences T' of $\mathcal{L}'$ is an *extension by definitions* of T over $\mathcal{L}$ when $T' = T \cup S$ and $S = \{\varphi_s \mid s \in \mathcal{L}' \setminus \mathcal{L}'\}$ and each φ_s is a definition of s in the language $\mathcal{L}$ over T.

Commonly used symbols such as $0, 1, \subseteq, \cap, \mathcal{P}, \bigcup$ are defined over ZF.

Theorem. Suppose that $\mathcal{L} \subseteq \mathcal{L}'$, and that T is a set of $\mathcal{L}$-sentences and T' is an extension by definitions of T to $\mathcal{L}'$. Then

(i) (conservativity) If φ is a sentence of $\mathcal{L}$, then $T \vdash \varphi \leftrightarrow T' \vdash \varphi$.

(ii) (abbreviations) If φ is a formula of $\mathcal{L}'$, then there exists a formula $\hat{\varphi}$ of $\mathcal{L}$ whose free variables are exactly those of φ, such that $T' \vdash \forall x. (\varphi \leftrightarrow \hat{\varphi})$.

Example. The intersection $a \cap b$ can be defined as the unique set c such that

$$\forall x. (x \in c \iff x \in a \wedge x \in b)$$

This definition makes sense only if there is a unique c satisfying this formula $\varphi(a, b, c)$. If

$$M = \{a, c, d, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$$

then it is easy to check that both $\{a\}$ and $\{a, b\}$ satisfy $\varphi(\{a, b, c\}, \{a, b, d\}, -)$, so intersection cannot be defined.

1.4 Absoluteness

It is often the case that definitions appear to give the same set regardless of which model we are working inside. For example, $\{x \mid x \neq x\}$ is the empty set in any model, and $\{x \mid x = a \vee x = b\}$ gives a pair set. Other definitions need not, for example $\mathcal{P}(\mathbb{N})$, which need not be the true power set in a given transitive model. To quantify this behaviour, we need to define what it means for φ to hold in an arbitrary structure; this concept is called *relativisation*.

Definition. The quantifier $\forall x \in a. \varphi$ is an abbreviation of $\forall x. (x \in a) \to \varphi$. Similarly, $\exists x \in a. \varphi$ is an abbreviation of $\exists x. (x \in a) \wedge \varphi$. Let W be a class; we define by recursion the *relativisation* φ^W of φ as follows.

$$\begin{aligned}
(x \in y)^W &\equiv x \in y \\
(x = y)^W &\equiv x = y \\
(\varphi \vee \psi)^W &\equiv \varphi^W \vee \psi^W \\
(\neg\varphi)^W &\equiv \neg\varphi^W \\
(\exists x. \varphi)^W &\equiv \exists x \in W. \varphi^W
\end{aligned}$$

One can easily show that

$$
\begin{aligned}
(\varphi \wedge \psi)^W &\equiv \varphi^W \wedge \psi^W \\
(\varphi \to \psi)^W &\equiv \varphi^W \to \psi^W \\
(\forall x.\, \varphi)^W &\equiv \forall x \in W.\, \varphi^W
\end{aligned}
$$

Proposition. Suppose that $M \subseteq N$ and M is a definable class over N. Then the relation $M \vDash \varphi$ is first-order expressible in N.

Proof. Suppose M is defined by θ, so

$$\forall z \in N.\, \theta(z) \leftrightarrow z \in M$$

We claim that $(N, \in) \vDash \varphi^M$ if and only if $(M, \in) \vDash \varphi$. We proceed by induction on the length of formulae. For example,

$$N \vDash (x \in y)^M \text{ iff } N \vDash x \in y \text{ and } x, y \in M \text{ iff } \theta(x), \theta(y), M \vDash x \in y$$

The case for equality is similar, and disjunction and negation are simple. Finally,

$$N \vDash (\exists x.\, \varphi(x))^M \text{ iff } N \vDash \exists x.\, x \in M \wedge \varphi^M(x)$$

which holds precisely when there is some $x \in N$ such that $N \vDash x \in M$ and $N \vDash \varphi^M(x)$, but $N \vDash x \in M$ if and only if $\theta(x)$, giving the result as required. □

Thus, relativisation is a way to express truth in definable classes.

Definition. Suppose that $M \subseteq N$ are classes and $\varphi(u_1, \dots, u_n)$ is a formula. Then φ is called

(i) *upwards absolute* for M, N if

$$\forall x_1, \dots, x_n \in M.\, (\varphi^M(x_1, \dots, x_n) \to \varphi^N(x_1, \dots, x_n))$$

(ii) *downwards absolute* for M, N if

$$\forall x_1, \dots, x_n \in M.\, (\varphi^N(x_1, \dots, x_n) \to \varphi^M(x_1, \dots, x_n))$$

(iii) *absolute* for M, N if it is both upwards and downwards absolute, or equivalently,

$$\forall x_1, \dots, x_n \in M.\, (\varphi^M(x_1, \dots, x_n) \leftrightarrow \varphi^N(x_1, \dots, x_n))$$

If $N = \mathrm{V}$, we simply say that φ is (upwards or downwards) absolute for M. If Γ is a set of formulas, we say that Γ is (upwards or downwards) absolute for M, N if and only if φ is (upwards or downwards) absolute for M, N for each $\varphi \in \Gamma$. Suppose T is a set of sentences and f is a defined function by φ. Then for $M \subseteq N$ models of T, we say that f is absolute for M, N precisely when φ is absolute for M, N.

Example. If $M \subseteq N$ both satisfy extensionality, then the empty set is absolute for M, N by the formula $\forall x \in a.\, (x \neq x)$. The power set of 2 is not absolute between 4 and V, because in 4, it has only two elements.

Example. $\varphi \leftrightarrow \psi$ does not imply $\varphi^M \leftrightarrow \psi^M$. Let $\varphi(v)$ be the statement $\forall x.\,(x \notin v)$; in ZF this defines $\varnothing$. Now, the following are two ways to express $0 \in z$.

$$\psi(z) \equiv \exists y.\,(\varphi(y) \wedge y \in z); \quad \theta(z) \equiv \forall y.\,(\varphi(y) \to y \in z)$$

Note that if there exists a unique y such that $\varphi(y)$, then these are equivalent. However, this is often not the case, for example if

$$a = 0; \quad b = \{0\}; \quad c = \{\{\{0\}\}\}; \quad M = \{a, b, c\}$$

then $\varphi^M(a)$ holds, so $\psi^M(b)$, but $\varphi^M(c)$ also holds, so $\theta^M(b)$ fails.

The main obstacle to absoluteness for basic statements turns out to be transitivity of the model.

Definition. Given classes $M \subseteq N$, we say that M is *transitive* in N if

$$\forall x, y \in N.\,(x \in M \wedge y \in x \to y \in M)$$

1.5 The Lévy hierarchy

Definition. The class of formulas Δ_0 is the smallest class Γ closed under the following conditions.

(i) if φ is atomic, $\varphi \in \Gamma$ (that is, $(v_i \in v_j) \in \Gamma$ and $(v_i = v_j) \in \Gamma$);
(ii) if $\varphi, \psi \in \Gamma$, then $\varphi \vee \psi \in \Gamma$ and $\neg\varphi \in \Gamma$; and
(iii) if $\varphi \in \Gamma$, then $(\forall v_i \in v_j.\,\varphi) \in \Gamma$ and $(\exists v_i \in v_j.\,\varphi) \in \Gamma$.

That is, Δ_0 is the class of formulas generated from atomic formulas by Boolean operations and bounded quantification.

Definition. We proceed by induction to define Σ_n and Π_n as follows.

(i) $\Sigma_0 = \Pi_0 = \Delta_0$;
(ii) if φ is Π_{n-1} then $\exists v_i.\,\varphi$ is Σ_n;
(iii) if φ is Σ_{n-1} then $\forall v_i.\,\varphi$ is Π_n.

Example. The formula $\forall v_1.\,\exists v_2.\,\forall v_3.\,(v_4 = v_3)$ is Π_3. But $(\forall v_1.\,v_1 = v_2) \wedge v_3 = v_4$ is not Π_n or Σ_n for any n.

Definition. Given an $\mathcal{L}_\in$-theory T, let Σ_n^T be the class of formulas Γ such that for any $\varphi \in \Gamma$, there exists $\psi \in \Sigma_n$ such that $T \vdash \varphi \leftrightarrow \psi$. We define Π_n^T analogously. A formula is in Δ_n^T if there exists $\psi \in \Sigma_n$ and $\theta \in \Pi_n$ such that $T \vdash \varphi \leftrightarrow \psi$ and $T \vdash \varphi \leftrightarrow \theta$.

Note that Δ_n only makes much sense with respect to some theory T for $n > 0$.

Lemma. If φ and ψ are in Σ_n^{ZF}, then so are

$$\exists v.\,\varphi; \quad \varphi \vee \psi; \quad \varphi \wedge \psi; \quad \exists v_i \in v_j.\,\varphi; \quad \forall v_i \in v_j.\,\varphi$$

If φ is in Σ_n^{ZF}, then $\neg\varphi$ is in Π_n^{ZF}. Further, for every φ, there exists n such that φ is in Σ_n^{ZF}, and if φ is in Σ_n^{ZF}, then φ is in Σ_m^{ZF} for all $m \geq n$.

Remark. $\exists x_1.\, \forall x_2.\, \exists x_3.\, \forall y.\, (y \in v \to v \neq v)$ is Σ_4, but is logically equivalent to the statement $\forall y \in v.\, v \neq v$, which is Σ_0. The fact that Σ_n^{ZF} is closed under bounded quantification depends on the axiom of collection. In particular, in Zermelo set theory, there is a Σ_1^{Z} formula φ such that $\forall x \in a.\, \varphi$ is not Σ_1^{Z}. In intuitionistic logic, these classes are very badly behaved; for instance, we could have a Π_1^T formula φ such that $\neg\varphi$ is not Σ_1^T.

We can now show absoluteness for Δ_0 formulas between transitive models.

Theorem. Let M be transitive in N and $M \subseteq N$, and let $\varphi(\mathbf{u})$ be a Δ_0-formula. Then, for any $\mathbf{a} \in M$,

$$M \vDash \varphi(\mathbf{a}) \text{ if and only if } N \vDash \varphi(\mathbf{a})$$

Proof. We prove this by induction on the class Δ_0. The cases of atomic formulas and propositional connectives are immediate, so it suffices to show the result for $\exists x \in a.\, \varphi$ where φ is absolute between M and N. Suppose $M \vDash \exists x \in a.\, \varphi(x)$, so there exists $b \in M$ such that $M \vDash b \in a \wedge \varphi(b)$. Then we also have $N \vDash b \in a \wedge \varphi(b)$ by absoluteness of φ, as required. Conversely, suppose $N \vDash \exists x \in a.\, \varphi(x)$, so there exists $b \in N$ such that $N \vDash b \in a \wedge \varphi(b)$. Since M is transitive in N, we obtain $b \in M$, so $M \vDash b \in a \wedge \varphi(b)$ by absoluteness of φ. $\square$

Proposition. The following are Δ_0^{ZF}, and therefore absolute between transitive models.

(i) $x \subseteq y$;
(ii) $a = \{x, y\}$ (the unordered pair);
(iii) $a = \langle x, y \rangle$ (the ordered pair);
(iv) $a = x \times y$;
(v) $a = \bigcup b$;
(vi) a is a transitive set;
(vii) $x = \varnothing$;
(viii) r is a relation;
(ix) r is a function;
(x) r is a relation with domain a and range b;
(xi) x is the pointwise image of r on a, denoted $r''a = \{y \mid \exists x \in a.\, \langle x, y \rangle \in r\}$;
(xii) $r|_a$.

Remark. The following are not absolute between transitive models, and thus not Δ_0^{ZF}.

(i) the cofinality function $\alpha \mapsto \mathrm{cf}(\alpha)$;

(ii) being a cardinal;

(iii) ω_1;

(iv) $y = \mathcal{P}(x)$.

Lemma. The statement that a given set a is finite is Δ_1^{ZF}.

Proposition. Let M be transitive in N and $M \subseteq N$. Then Σ_1 formulas are upwards absolute between M and N, and Π_1 formulas are downwards absolute between M and N.

Corollary. Δ_1^{ZF} formulas are absolute between transitive models.

Lemma. (ZF) The statement that α is an ordinal is absolute.

Proof. First, note that α is an ordinal in ZF if and only if it is a transitive set of transitive sets. This can be written as

$$(\forall \beta \in \alpha. \forall \gamma \in \beta. \gamma \in \alpha) \wedge (\forall \beta \in \alpha. \forall \gamma \in \beta. \forall \delta \in \gamma. \delta \in \beta)$$

which is Δ_0, as required. □

We can give a slightly better rephrasing of this lemma.

Lemma. The statement that r is a strict total ordering of a is Δ_0.

Proof. The statement that r is a transitive relation on a is that

$$\forall xyz \in a. (\langle x, y \rangle \in r \wedge \langle y, z \rangle \in r \to \langle x, z \rangle \in r)$$

Trichotomy is

$$\forall xy \in a. (\langle x, y \rangle \in r \vee \langle y, x \rangle \in r \vee x = y)$$

Irreflexivity is

$$\forall x \in a. \langle x, x \rangle \notin r$$

□

Corollary. The statement that x is a transitive set totally ordered by $\in$ is Δ_0, and thus being an ordinal is Δ_0.

Lemma. (ZF) The statement that r is well-founded on a is Δ_1^{ZF}.

Proof. The Π_1 formula is

$$r \text{ is a relation on } a \wedge [\forall X. (\exists x \in X. (z = z) \wedge X \subseteq a) \to \exists x \in X. \forall y \in X. \langle y, z \rangle \notin r]$$

For the Σ_1 formula, we first show that a relation is well-founded on a if and only if there exists a function $a \to \mathrm{Ord}$ such that $\langle y, x \rangle \in r$ implies $f(y) \in f(x)$. Suppose r is well-founded; we then define $f : a \to \mathrm{Ord}$ by $f(x) = \sup\{f(y) + 1 \mid \langle y, x \rangle \in r\}$, and one can show that this satisfies the required property. For the other direction, let $X \subseteq a$ be a nonempty subset, and consider the pointwise image

$f''X$. This has a minimal element α, then for any $z \in X$, if $f(z) = \alpha$ then for all $y \in X$, we have $f(y) \geq \alpha$, so $\langle y, z\rangle \notin r$. We then define well-foundedness with a Σ_1 formula as follows.

$$\exists f. (f \text{ is a function} \wedge \forall u \in \operatorname{ran} f. (u \in \mathrm{Ord}) \wedge \forall xy \in a. (\langle y, x\rangle \in r \to f(y) \in f(x)))$$

□

Proposition. The following are Δ_0^{ZF}.
(i) x is a limit ordinal;
(ii) x is a successor ordinal;
(iii) x is a finite ordinal;
(iv) $x = \omega$;
(v) $x = n$ for any finite ordinal n.

Proposition. The following are Π_1^{ZF} and hence downwards absolute between transitive models.
(i) κ is a cardinal;
(ii) κ is regular;
(iii) κ is a limit cardinal;
(iv) κ is a strong limit cardinal.

Lemma. (ZF) Let W be a nonempty transitive class. Then the axioms of extensionality, empty set, and foundation all hold in W.

Proof. For extensionality, the relativisation of

$$\forall x. \forall y. (\forall z. (z \in x \leftrightarrow x \in y) \to x = y)$$

to W is

$$\forall x \in W. \forall y \in W. (\forall z \in W. (z \in x \leftrightarrow x \in y) \to x = y)$$

Suppose $x \in W, y \in W$, but $x \neq y$. Then by extensionality in the metatheory, without loss of generality we can fix $z \in x$ with $z \notin y$. But since W is transitive, we must have $z \in W$, contradicting $x = y$, as required.

As W is nonempty, we can use foundation to fix $x \in W$ such that $x \cap W = \varnothing$. Since W is transitive, $x \subseteq W$, and therefore $x = \varnothing \in W$. Moreover, the statement that $x = \varnothing$ is Δ_0 and therefore absolute.

□

Lemma. (ZF) Let W be a transitive class. Then
(i) if for any pair $x, y \in W$, the real pair set $\{x, y\}$ lies in W, then the axiom of pairing holds in W;
(ii) if for any set $x \in W$, the union $\bigcup x$ lies in W, then the axiom of union holds in W;
(iii) if $\omega \in W$, then the axiom of infinity holds in W;
(iv) if, for every formula φ with free variables in $\{x, a, v_1, \dots, v_n\}$, we have

$$\forall a, v_1, \dots, v_n \in W. \{x \in a \mid \varphi^W(x, a, v_1, \dots, v_n)\} \in W$$

then the axiom of separation holds in W;

(v) if, for every formula φ with free variables in $\{x, y, a, v_1, \dots, v_n\}$, for all $a, v_1, \dots, v_n \in W$, if

$$\forall x \in a.\, \exists! y \in W.\, \varphi^W(x, y, a, v_1, \dots, v_n)$$

then

$$\exists b \in W.\, \{y \mid \exists x \in a.\, \varphi^W(x, y, a, v_1, \dots, v_n)\} \subseteq b$$

then the axiom of replacement holds in W;

(vi) if, for every $a \in W$, there exists $b \in W$ such that $\mathcal{P}(a) \cap W = b$, then the axiom of power set holds in W.

Corollary. (ZF) If W is a nonempty transitive class satisfying the conditions of the previous lemma, it is a model of ZF.

1.6 Transfinite recursion

Definition. A relation R is *set-like* on a class A if for all $x \in A$, the collection of R-predecessors of x is a set.

Example. $\in$ is set-like on V, but $\ni$ is not set-like on V.

Let A be a class, and let φ be such that $A = \{x \mid \varphi(x)\}$. Then $A^W = \{x \mid \varphi^W(x)\}$. We say that A is absolute for W if $A^W = A \cap W$. Viewing a class relation $R \subseteq \mathrm{V} \times \mathrm{V}$ as a collection of ordered pairs $\{\langle x, y\rangle \mid \psi(x, y)\}$, we have $R^W = \{\langle x, y\rangle \mid \psi^W(x, y)\}$, and say that R is absolute for W if $R^W = R \cap W^2$. Observe that if R is a class function, we can only refer to the *function* R^W if we first check that $(\forall x.\, \exists! y.\, \varphi(x, y))^W$. In this case, we have $R^W : W \to W$, and we say that R is an absolute function for W iff $R^W = R|_W$.

We briefly recall the transfinite recursion theorem.

Theorem. Let R be a relation which is well-founded and set-like on a class A. Let $F : A \times \mathrm{V} \to \mathrm{V}$ be a class function. Given $x \in A$, let $\mathrm{pred}(A, x, R) = \{y \in A \mid y \mathrel{R} x\}$ be the set of R-predecessors of x in A. Then there is a unique function $G : A \to \mathrm{V}$ such that for all $x \in A$,

$$G(x) = F\left(x, G\Big|_{\mathrm{pred}(A, x, R)}\right)$$

We now prove the absoluteness of transfinite recursion.

Theorem. Let R be a relation which is well-founded and set-like on a class A. Let $F : A \times \mathrm{V} \to \mathrm{V}$ be a class function, and let $G : A \to \mathrm{V}$ be the unique function given by applying transfinite recursion to F. Suppose that W is a transitive model of ZF, and suppose that the following hold.

(i) A and F are absolute for W;

(ii) R is absolute for W and (R is set-like on A)^W;

(iii) for all $x \in W$, $\mathrm{pred}(A, x, R) \subseteq W$.

Then G is absolute for W.

Proof. By absoluteness, $A^W = A \cap W$ and $R^W = R \cap W^2$. Hence, every nonempty subset of A^W has an R^W-minimal element. In particular, (R is well-founded on A)^W. We can then apply transfinite recursion in W to define a unique function $G^W : A^W \to W$ such that for all $x \in A^W$,

$$G^W(x) = F^W\left(x, G^W\Big|_{\mathrm{pred}^W(A^W, x, R^W)}\right)$$

To prove absoluteness for G, it suffices to show that $G^W = G|_{A^W}$. We show this by transfinite induction in W. Suppose that for all $y \ R \ x$, we have $G^W(y) = G(y)$. By absoluteness, (iii), and the inductive hypothesis, we obtain

$$G^W(x) = F^W\left(x, G^W\Big|_{\mathrm{pred}^W(A^W, x, R^W)}\right) = F\left(x, G\Big|_{\mathrm{pred}(A, x, R)}\right) = G(x)$$

□

Corollary. The following are absolute for transitive models of ZFC:

(i) the rank function;

(ii) the transitive closure of a set;

(iii) the addition and multiplication operations of ordinal arithmetic.

1.7 The reflection theorem

In this subsection, we will not use choice.

Recall the Tarski–Vaught test: if $\mathcal{M}$ is a substructure of $\mathcal{N}$ with universes M and N respectively, then the following two statements are equivalent.

(i) $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$;

(ii) for any formula $\varphi(v, \mathbf{w})$ and $\mathbf{a} \in M$, if there exists $b \in N$ such that $\mathcal{N} \vDash \varphi(b, \mathbf{a})$, then there exists $c \in M$ such that $\mathcal{M} \vDash \varphi(c, \mathbf{a})$.

Definition. A finite list of formulas $\boldsymbol{\varphi} = \varphi_1, \dots, \varphi_n$ is said to be *subformula closed* if every subformula of the φ_i is contained on the list.

We can now state a version of the Tarski–Vaught test for classes.

Lemma. Let $\boldsymbol{\varphi}$ be a subformula closed list of formulas, and suppose $W \subseteq Z$ are nonempty classes. Then the following two statements are equivalent.

(i) each formula in $\boldsymbol{\varphi}$ is absolute for W and Z;

(ii) whenever φ_i is of the form $\exists x.\, \varphi_j(x, \mathbf{y})$ where the free variables of φ_j are equal to x or contained in $\mathbf{y}$, then

$$\forall \mathbf{y} \in W. \left(\exists x \in Z.\, \varphi_j^Z(x, \mathbf{y}) \to \exists x \in W.\, \varphi_j^Z(x, \mathbf{y})\right)$$

Proof. *(i) implies (ii).* Suppose that each formula in $\boldsymbol{\varphi}$ is absolute. Let φ_i be of the form $\exists x.\, \varphi_j(x, \mathbf{y})$, and fix $\mathbf{y} \in W$. Then $\varphi_i^Z(\mathbf{y})$ is $\exists x \in Z.\, \varphi_j^Z(x, \mathbf{y})$. If this holds, by absoluteness $\varphi_i^W(\mathbf{y})$ holds, so there is $x \in W$ such that $\varphi_j^W(x, \mathbf{y})$. Finally, $W \subseteq Z$ and absoluteness of φ_j gives $\exists x \in W.\, \varphi_j^Z(x, \mathbf{y})$.

(ii) implies (i). We show this by induction on the length of φ_i. The result if φ_i is atomic or of the form $\varphi_j \vee \varphi_k$ or $\neg\varphi_j$ is immediate. Suppose φ_i is of the form $\exists x.\, \varphi_j(x, \mathbf{y})$, and fix $\mathbf{y} \in W$. Then $\varphi_i^Z(\mathbf{y})$ is equivalent to the statement $\exists x \in Z.\, \varphi_j^Z(x, \mathbf{y})$. By (ii), this gives $\exists x \in W.\, \varphi_j^Z(x, \mathbf{y})$. Since $W \subseteq Z$, the reverse implication is trivial. But $\exists x \in W.\, \varphi_j^Z(x, \mathbf{y})$ is equivalent to the statement that $\varphi_i^W(\mathbf{y})$ holds, as required. $\square$

Theorem (reflection theorem)**.** Let W be a nonempty class, and suppose that there is a class function F_W such that for any ordinal α, $F_W(\alpha) = W_\alpha \in \mathrm{V}$. Suppose that
(i) if $\alpha < \beta$, then $W_\alpha \subseteq W_\beta$;
(ii) if λ is a limit ordinal, then $W_\lambda = \bigcup_{\alpha<\lambda} W_\alpha$;
(iii) $W = \bigcup_{\alpha\in\mathrm{Ord}} W_\alpha$.
Then for any finite list of formulas $\boldsymbol{\varphi} = \varphi_1, \dots, \varphi_n$, ZF proves that for every α there is a limit ordinal $\beta > \alpha$ such that the φ_i are absolute between W_β and W.

One example of such a class function is $W_\alpha = \mathrm{V}_\alpha$.

Corollary (Montague–Lévy reflection)**.** For any finite list of formulas $\boldsymbol{\varphi} = \varphi_1, \dots, \varphi_n$, ZF proves that for every α there is a limit ordinal $\beta > \alpha$ such that the φ_i are absolute for V_β.

We now prove the reflection theorem.

Proof. Let $\boldsymbol{\varphi} = \varphi_1, \dots, \varphi_n$ be a finite list of formulas. By extending the list and taking logical equivalences if necessary, we will assume that this list is subformula-closed and that there are no universal quantifiers. For $i \leq n$, we will define a function $G_i : \mathrm{Ord} \to \mathrm{Ord}$ as follows. If φ_i is of the form $\exists x.\, \varphi_j(x, \mathbf{y})$ where $\mathbf{y}$ is a tuple of length k_i, we will define a function $F_i : W^{k_i} \to \mathrm{Ord}$ by setting

$$F_i(\mathbf{y}) = \begin{cases} 0 & \text{if } \neg\exists x \in W.\, \varphi_j^W(x, \mathbf{y}) \\ \eta & \text{where } \eta \text{ is the least ordinal such that } \exists x \in W_\eta.\, \varphi_j^W(x, \mathbf{y}) \end{cases}$$

We set
$$G_i(\delta) = \sup\left\{F_i(\mathbf{y}) \mid \mathbf{y} \in W_\delta^{k_i}\right\}$$

If φ_i is not of this form, we set $G_i(\delta) = 0$ for all δ. Finally, we let

$$K(\delta) = \max\{\delta + 1, G_1(\delta), \dots, G_n(\delta)\}$$

Note that the F_i work in an analogous way to Skolem functions, but does not require choice. The F_i are well-defined, and, using replacement in V, since W_δ is a set, $F_i'' W_\delta^{k_i}$ is also a set in V, so G_i and K are both defined and take values in Ord. Also, G_i is monotone: if $\delta \leq \delta'$ then $G_i(\delta) \leq G(\delta')$.

We claim that for every α there is a limit ordinal $\beta > \alpha$ such that for all $\delta < \beta$ and $i \leq n$, we have $G_i(\delta) < \beta$; that is, β is closed under this process of finding witnesses. Set $\lambda_0 = \alpha$ and let $\lambda_{t+1} = K(\lambda_t)$. Then we set $\beta = \sup_{t\in\omega} \lambda_t$, which is a limit ordinal as it is the supremum of a strictly increasing sequence of ordinals. If $\delta < \beta$, then $\delta < \lambda_t$ for some t, so $G_i(\delta) \leq G_i(\lambda_t)$ by monotonicity, but $G_i(\lambda_t) \leq K(\lambda_t) = \lambda_{t+1} < \beta$ as required.

To complete the theorem, it suffices to consider φ_i of the form $\exists x.\, \varphi_j(x, \mathbf{y})$ by the Tarski–Vaught test for classes above. Fix $\mathbf{y} \in W_\beta$, and suppose there exists $x \in W$ such that $\varphi_j^W(x, \mathbf{y})$. Since β is a limit ordinal and $\mathbf{y}$ is a finite sequence in W_β, we must have $\mathbf{y} \in W_\gamma$ for some $\gamma < \beta$. Thus

$$0 < F_i(\mathbf{y}) \leq G_i(\gamma) < \beta$$

so by construction, there exists a witness $x \in W_\beta$ such that $\varphi_j^W(x, \mathbf{y})$. Hence $\boldsymbol{\varphi}$ is absolute between W_β and W as required. $\square$

Remark. This is a theorem scheme; for every choice of formulas $\boldsymbol{\varphi}$, it is a theorem of ZF that $\boldsymbol{\varphi}$ are absolute for some V_β. We cannot prove that for every collection of formulas $\boldsymbol{\varphi}$, for all ordinals α there exists $\beta > \alpha$ such that $\boldsymbol{\varphi}$ is absolute for W_β, W. Note that even if $\boldsymbol{\varphi}$ is absolute for W_β and W, we need not have $\boldsymbol{\varphi}^{W_\beta}$.

If $\boldsymbol{\varphi}$ is any finite list of axioms of ZF, then there are arbitrarily large β such that $\boldsymbol{\varphi}$ holds in V_β. If β is a limit ordinal, $\mathrm{V}_\beta \vDash \mathrm{Z(C)}$, so we may restrict our attention to instances of replacement.

Corollary. Let T be an extension of ZF in $\mathcal{L}_\in$, and let $\varphi_1, \dots, \varphi_n$ be a finite list of axioms from T. Then T proves that for every α there exists $\beta > \alpha$ such that $\left(\bigwedge_{i=1}^n \varphi_i\right)^{\mathrm{V}_\beta}$.

Corollary. (ZFC) Let W be a class and let $\varphi_1, \dots, \varphi_n$ be a finite list of formulas in $\mathcal{L}_\in$. Then ZFC proves that for every transitive $x \subseteq W$, there exists some transitive $y \supseteq x$ such that the φ_i are absolute between y and W, and $|y| \leq \max\{w, |x|\}$.

Taking $x = \omega$ and $W = \mathrm{V}$ gives the following result.

Corollary. Let T be any set of sentences in $\mathcal{L}_\in$ such that $T \vdash \mathrm{ZFC}$. Let $\varphi_i, \dots, \varphi_n \in T$. Then T proves that there is a transitive set y of cardinality $\aleph_0$ such that $\left(\bigwedge_{i=1}^n \varphi_i\right)^y$.

Corollary. Let T be any consistent set of sentences in $\mathcal{L}_\in$ such that $T \vdash \mathrm{ZF}$. Then T is not finitely axiomatisable. That is, for any finite set of sentences Γ in $\mathcal{L}_\in$ such that $T \vdash \Gamma$, there exists a sentence φ such that $T \vdash \varphi$ but $\Gamma \nvdash \varphi$.

This only holds for first-order theories without classes; for example, Gödel–Bernays set theory is finitely axiomatisable.

Proof. Let $\varphi_1, \dots, \varphi_n$ be a set of sentences such that $T \vdash \bigwedge_{i=1}^n \varphi_i$. Suppose that $\bigwedge_{i=1}^n \varphi_i$ proves every axiom of T. By reflection, T proves that for every α there is $\beta > \alpha$ such that the φ_i hold in V_β if and only if they hold in V. Since they hold in V, they must hold in some V_β. Fix β_0 to be the least ordinal such that $\bigwedge_{i=1}^n \varphi_i^{\mathrm{V}_{\beta_0}}$. Then all of the axioms of T hold in V_{β_0}, so $\mathrm{V}_{\beta_0} \vDash T$. Since T extends ZF, our basic absoluteness results hold, so in particular, if $\alpha \in \mathrm{V}_{\beta_0}$ then

$$\mathrm{V}_\alpha^{\mathrm{V}_{\beta_0}} = \mathrm{V}_\alpha \cap \mathrm{V}_{\beta_0} = \mathrm{V}_\alpha$$

So V_α is absolute for V_{β_0}. Note that T proves that there exists α such that $\bigwedge_{i=1}^n \varphi_i^{V_\alpha}$, but as V_{β_0} satisfies every axiom of T, this must be true in V_{β_0}. That is, there must be $\alpha < \beta_0$ such that $\bigwedge_{i=1}^n \varphi_i^{V_\alpha}$. This contradicts minimality of β_0. □

1.8 Cardinal arithmetic

In this subsection, we will use the axiom of choice. We recall the following basic definitions and results.

Definition. The *cardinality* of a set x, written $|x|$, is the least ordinal α such that there is a bijection $x \to \alpha$.

This definition only makes sense given the well-ordering principle.

Definition. The cardinal arithmetic operations are defined as follows. Let κ, λ be cardinals.

(i) $\kappa + \lambda = |\{0\} \times \kappa \cup \{1\} \times \lambda|$;
(ii) $\kappa \cdot \lambda = |\kappa \times \lambda|$;
(iii) $\kappa^\lambda = |\kappa^\lambda|$, the cardinality of the set of functions $\lambda \to \kappa$;
(iv) $\kappa^{<\lambda} = \sup\{\kappa^\alpha \mid \alpha < \lambda, \alpha \text{ a cardinal}\}$.

Theorem (Hessenberg)**.** If κ, λ are infinite cardinals, then

$$\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$$

Lemma. If κ, λ, μ are cardinals, then

$$\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu; \quad (\kappa^\lambda)^\mu = \kappa^{\lambda\cdot\mu}$$

Definition. A map between ordinals $\alpha \to \beta$ is *cofinal* if $\sup \operatorname{ran} f = \beta$. The *cofinality* of an ordinal γ, written $\mathrm{cf}(\gamma)$, is the least ordinal that admits a cofinal map to γ. A limit ordinal γ is *singular* if $\mathrm{cf}(\gamma) < \gamma$, and *regular* if $\mathrm{cf}(\gamma) = \gamma$.

Remark. (i) Since the identity map is always cofinal, we have $\mathrm{cf}(\gamma) \leq \gamma$.

(ii) $\omega = \mathrm{cf}(\omega) = \mathrm{cf}(\omega + \omega) = \mathrm{cf}(\aleph_\omega)$.

(iii) $\mathrm{cf}(\gamma) \leq |\gamma|$.

Theorem. Let γ be a limit ordinal. Then

(i) if γ is regular, γ is a cardinal;
(ii) the cardinal successor γ^+ is a regular cardinal;
(iii) $\mathrm{cf}(\mathrm{cf}(\gamma)) = \mathrm{cf}(\gamma)$, so $\mathrm{cf}(\gamma)$ is regular;
(iv) $\aleph_\alpha$ is regular whenever $\alpha = 0$ or a successor;
(v) if λ is a limit ordinal, $\mathrm{cf}(\aleph_\lambda) = \mathrm{cf}(\lambda)$.

Theorem. Let κ be a regular cardinal. If $\mathcal{F}$ is a family of sets with $|\mathcal{F}| < \kappa$ and each $|X| < \kappa$ for $X \in \mathcal{F}$, then $|\bigcup \mathcal{F}| < \kappa$.

Proof. We show this by induction on $|\mathcal{F}| = \gamma < \kappa$. Suppose the claim holds for γ, and consider $\mathcal{F} = \{X_\alpha \mid \alpha < \gamma + 1\}$. Then, assuming the sets involved are infinite,

$$\left|\bigcup \mathcal{F}\right| = \left|\bigcup_{\alpha<\gamma} X_\alpha \cup X_\gamma\right| = \left|\bigcup_{\alpha<\gamma} X_\alpha\right| + |X_\gamma| = \max\left\{\left|\bigcup_{\alpha<\gamma} X_\alpha\right|, |X_\gamma|\right\} < \kappa$$

Now suppose γ is a limit, and suppose the claim holds for all $\beta < \gamma$. Let $\mathcal{F} = \{X_\alpha \mid \alpha < \gamma\}$, and define $g : \gamma \to \kappa$ by

$$g(\beta) = \left|\bigcup_{\alpha<\beta} X_\beta\right|$$

But κ is regular and $\gamma < \kappa$, so this map is not cofinal. Hence $g''\gamma = |\bigcup \mathcal{F}| < \kappa$. □

We can generalise the notions of cardinal sum and product as follows.

Definition. Let $(\kappa_i)_{i\in I}$ be an indexed sequence of cardinals, and let $(X_i)_{i\in I}$ be a sequence of pairwise disjoint sets with $|X_i| = \kappa_i$ for all $i \in I$. Then the *cardinal sum* of (κ_i) is

$$\sum_{i\in I} \kappa_i = \left|\bigcup_{i\in I} X_i\right|$$

The *cardinal product* is

$$\prod_{i\in I} \kappa_i = \left|\prod_{i\in I} X_i\right|$$

where $\prod_{i\in I} X_i$ denotes the set of functions $f : I \to \bigcup_{i\in I} X_i$ such that $f(i) \in X_i$ for each i.

The following theorem generalises Cantor's diagonal argument.

Theorem (König's theorem)**.** Let I be an indexing set, and suppose that $\kappa_i < \lambda_i$ for all $i \in I$. Then

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$$

Proof. Let $(B_i)_{i\in I}$ be a sequence of disjoint sets with $|B_i| = \lambda_i$, and let $B = \prod_{i\in I} B_i$. It suffices to show that for any sequence $(A_i)_{i\in I}$ of subsets of B such that for all $i \in I$, $|A_i| = \kappa_i$, then

$$\bigcup_{i\in I} A_i \neq B$$

Given such a sequence, we let S_i be the projection of A_i onto its ith coordinate.

$$S_i = \{f(i) \mid f \in A_i\}$$

Then by definition, $S_i \subseteq B_i$, and

$$|S_i| \leq |A_i| = \kappa_i < \lambda_i = |B_i|$$

Fix $t_i \in B_i \setminus S_i$. Finally, we define $g \in B$ by $g(i) = t_i$; by construction, we have $g \notin A_i$ for all i, so $g \in B$ but $g \notin \bigcup_{i \in I} A_i$. □

Corollary. If $\kappa \geq 2$ and λ is infinite, then

$$\kappa^\lambda > \lambda$$

Proof.

$$\lambda = \sum_{\alpha<\lambda} 1 < \prod_{\alpha<\lambda} 2 = 2^\lambda \leq \kappa^\lambda$$

□

Corollary. $\mathrm{cf}(2^\lambda) > \lambda$.

Proof. Let $f : \lambda \to 2^\lambda$, we show that $|\bigcup f''\lambda| < 2^\lambda$. Since for all $i \in I$, we have $f(i) < 2^\lambda$, we deduce

$$\left|\bigcup f''\lambda\right| \leq \sum_{i<\lambda} |f(i)| < \prod_{i<\lambda} 2^\lambda = (2^\lambda)^\lambda = 2^{\lambda\cdot\lambda} = 2^\lambda$$

□

Corollary. $2^{\aleph_0} \neq \kappa$ for any κ of cofinality $\aleph_0$. In particular, $2^{\aleph_0} \neq \aleph_\omega$.

Corollary. $\kappa^{\mathrm{cf}(\kappa)} > \kappa$ for every infinite cardinal κ.

We can prove very little in general about cardinal exponentiation given ZFC.

Definition. The *generalised continuum hypothesis* is the statement that $2^\kappa = \kappa^+$ for every infinite cardinal κ. Equivalently, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

Under this assumption, we can show the following.

Theorem. (ZFC + GCH) Let κ, λ be infinite cardinals.
(i) if $\kappa < \lambda$, then $\kappa^\lambda = \lambda^+$;
(ii) if $\mathrm{cf}(\kappa) \leq \lambda < \kappa$, then $\kappa^\lambda = \kappa^+$;
(iii) if $\lambda < \mathrm{cf}(\kappa)$, then $\kappa^\lambda = \kappa$.

When we construct models with certain properties of cardinal arithmetic, we will often want to start with a model satisfying GCH so that we have full control over cardinal exponentiation. Without this assumption, we know much less. The following theorems are essentially the only restrictions that we have on regular cardinals that are provable in ZFC.

Theorem. Let κ, λ be cardinals. Then
(i) if $\kappa < \lambda$, then $2^\kappa \leq 2^\lambda$;
(ii) $\mathrm{cf}(2^\kappa) > \kappa$;
(iii) if κ is a limit cardinal, then $2^\kappa = (2^{<\kappa})^{\mathrm{cf}(\kappa)}$.

Theorem. Let κ, λ be infinite cardinals. Then
(i) if $\kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$;
(ii) if $\mu < \kappa$ is such that $\mu^\lambda \geq \kappa$, then $\kappa^\lambda = \mu^\lambda$;
(iii) if $\kappa > \lambda$ and $\mu^\lambda < \kappa$ for all $\mu < \kappa$, then
  (a) if $\mathrm{cf}(\kappa) > \lambda$, then $\kappa^\lambda = \kappa$;
  (b) if $\mathrm{cf}(\kappa) \leq \lambda$, then $\kappa^\lambda = \kappa^{\mathrm{cf}(\kappa)}$.

Theorem (Silver)**.** Suppose that κ is a singular cardinal such that $\mathrm{cf}(\kappa) > \aleph_0$ and $2^\alpha = \alpha^+$ for all $\alpha < \kappa$. Then $2^\kappa = \kappa^+$.

This theorem therefore states that the generalised continuum hypothesis cannot first break at a singular cardinal with cofinality larger than $\aleph_0$.

Remark. It is consistent (relative to large cardinals, such as a measurable cardinal) to have $2^{\aleph_n} = \aleph_{n+1}$ for all $n \in \omega$, but $2^{\aleph_\omega} = \aleph_{\omega+2}$.

Theorem (Shelah)**.** Suppose that $2^{\aleph_n} < \aleph_\omega$ for all $n \in \omega$, so $\aleph_\omega$ is a strong limit cardinal. Then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

It is not known if this bound can be improved, but it is conjectured that $2^{\aleph_\omega} < \aleph_{\omega_1}$.

2 Constructibility

In this section, we will prove
$$\mathsf{Con}(\mathsf{ZF}) \to \mathsf{Con}(\mathsf{ZFC} + \mathsf{GCH})$$

2.1 Definable sets

Recall that the V_α hierarchy has the property that $\mathrm{V}_{\alpha+1} = \mathcal{P}(\mathrm{V}_\alpha)$. We will construct a universe L in which we restrict to the 'nice' subsets.

Definition. A set x is said to be *definable* over $(M, \in)$ if there exist $a_1, \dots, a_n \in M$ and a formula φ such that
$$x = \{z \in M \mid (M, \in) \vDash \varphi(z, a_1, \dots, a_n)\}$$
We write
$$\mathrm{Def}(M) = \{x \subseteq M \mid x \text{ is definable over } M\}$$

Remark. (i) $M \in \mathrm{Def}(M)$.
(ii) $M \subseteq \mathrm{Def}(M) \subseteq \mathcal{P}(M)$.

This definition involves a quantification over infinitely many formulas, so is not yet fully formalised. One method to do this is to code formulas as elements of V_ω, called *Gödel codes*. We can then use Tarski's *satisfaction relation* to define a formula Sat, and can then prove

$$\mathsf{Sat}(M, \in, \ulcorner\varphi\urcorner, x_1, \dots, x_n) \leftrightarrow (M, \in) \vDash \varphi(x_1, \dots, x_n)$$

where $\ulcorner\varphi\urcorner \in \mathrm{V}_\omega$ is the Gödel code for φ. We will later use a different method to formalise it, but for now we will assume that this is well-defined.

2.2 Defining the constructible universe

We define the L_α hierarchy by transfinite recursion as follows.

$$\mathrm{L}_0 = \varnothing; \quad \mathrm{L}_{\alpha+1} = \mathrm{Def}(\mathrm{L}_\alpha); \quad \mathrm{L}_\lambda = \bigcup_{\alpha<\lambda} \mathrm{L}_\alpha; \quad \mathrm{L} = \bigcup_{\alpha\in\mathrm{Ord}} \mathrm{L}_\alpha$$

Lemma. For any ordinals α, β,

(i) if $\beta \leq \alpha$ then $\mathrm{L}_\beta \subseteq \mathrm{L}_\alpha$;
(ii) if $\beta < \alpha$ then $\mathrm{L}_\beta \in \mathrm{L}_\alpha$;
(iii) L_α is transitive;
(iv) the ordinals of L_α are precisely α;
(v) L is transitive and $\mathrm{Ord} \subseteq \mathrm{L}$.

Definition. Let T be a set of axioms in $\mathcal{L}_\in$, and let W be a class. Then W is called an *inner model* of T if

(i) W is a transitive class;
(ii) $\mathrm{Ord} \subseteq W$;
(iii) T^W is true; that is, for every formula φ in T, we have φ^W.

Theorem. L is an inner model of ZF.

This is a theorem scheme; for every axiom of ZF, we can prove its relativisation to L.

Proof. By the previous lemma, it suffices to check that ZF^{L} holds.

- Since L is transitive, L satisfies extensionality and foundation.
- For the axiom of empty set, we use the fact that $\varnothing^{\mathrm{L}} = \varnothing = \mathrm{L}_0 \in \mathrm{L}$.
- For pairing, given $a, b \in \mathrm{L}$, we must show $\{a, b\} \in \mathrm{L}$. Fix α such that $a, b \in \mathrm{L}_\alpha$. Then

$$\{a, b\} = \{x \in \mathrm{L}_\alpha \mid (\mathrm{L}_\alpha, \in) \vDash x = a \vee x = b\} \in \mathrm{Def}(\mathrm{L}_\alpha)$$

- For union, let $a \in \mathrm{L}_\alpha$. By transitivity, $\bigcup a \subseteq \mathrm{L}_\alpha$. Then

$$\bigcup a = \{x \in \mathrm{L}_\alpha \mid (\mathrm{L}_\alpha, \in) \vDash \exists z.\, (z \in a \wedge x \in z)\} \in \mathrm{Def}(\mathrm{L}_\alpha)$$

- For infinity, note that

$$\omega = \{n \in \mathrm{L}_\omega \mid (\mathrm{L}_\omega, \in) \vDash n \in \mathrm{Ord}\} \in \mathrm{Def}(\mathrm{L}_\omega)$$

- Consider separation. Let φ be a formula, and let $a, \mathbf{u} \in \mathrm{L}_\alpha$. We claim that

$$b = \left\{x \in a \mid \varphi^{\mathrm{L}}(x, \mathbf{u})\right\} \in \mathrm{L}$$

  This implicitly uses the fact that L is definable. Using the reflection theorem, there is $\beta > \alpha$ such that

$$\mathsf{ZF} \vdash \forall x \in \mathrm{L}_\beta. (\varphi^{\mathrm{L}}(x, \mathbf{u}) \leftrightarrow \varphi^{\mathrm{L}_\beta}(x, \mathbf{u}))$$

  Moreover, $\varphi^{\mathrm{L}_\beta}(x, \mathbf{u})$ holds if and only if $(\mathrm{L}_\beta, \in) \vDash \varphi(x, \mathbf{u})$. We thus obtain

$$\left\{x \in a \mid \varphi^{\mathrm{L}}(x, \mathbf{u})\right\} = \left\{x \in a \mid \varphi^{\mathrm{L}_\beta}(x, \mathbf{u})\right\} = \left\{x \in \mathrm{L}_\beta \mid (\mathrm{L}_\beta, \in) \vDash \varphi(x, \mathbf{u}) \wedge x \in a\right\} \in \mathrm{Def}(\mathrm{L}_\beta)$$

- We now consider replacement. It suffices to show that if $a \in \mathrm{L}$ and $f : a \to \mathrm{L}$ is a definable function, then there exists $\gamma \in \mathrm{Ord}$ such that $f''a \subseteq \mathrm{L}_\gamma$, since then we can use separation. First, observe that for every $x \in a$, there exists $\beta \in \mathrm{Ord}$ such that $f(x) \in \mathrm{L}_\beta$. Using replacement in V, there exists an ordinal γ such that for all $x \in a$, there exists $\beta < \gamma$ such that $f(x) \in \mathrm{L}_\beta$. As $\mathrm{L}_\beta \subseteq \mathrm{L}_\gamma$, we thus obtain for all $x \in a$ that $f(x) \in \mathrm{L}_\gamma$.
- Finally, consider the axiom of power set. It suffices to prove that if $x \in \mathrm{L}$ then $\mathcal{P}(x) \cap \mathrm{L} \in \mathrm{L}$. Take $x \in \mathrm{L}$. Using replacement in V, we can fix an ordinal γ such that $\mathcal{P}(x) \cap \mathrm{L} \subseteq \mathrm{L}_\gamma$. Then

$$\mathcal{P}(x) \cap \mathrm{L} = \left\{z \in \mathrm{L}_\gamma \mid (\mathrm{L}, \in) \vDash z \subseteq x\right\} \in \mathrm{Def}(\mathrm{L}_\gamma)$$

□

2.3 Gödel functions

We will now formally define L. For clarity, we will define the ordered triple $\langle a, b, c\rangle$ to be $\langle a, \langle b, c\rangle\rangle$.

Definition. The *Gödel functions* are the following collection of functions on two variables.

(i) $\mathcal{F}_1(x, y) = \{x, y\}$;
(ii) $\mathcal{F}_2(x, y) = \bigcup x$;
(iii) $\mathcal{F}_3(x, y) = x \setminus y$;
(iv) $\mathcal{F}_4(x, y) = x \times y$;
(v) $\mathcal{F}_5(x, y) = \operatorname{dom} x = \{\pi_1(z) \mid z \in x \wedge z \text{ is an ordered pair}\}$;
(vi) $\mathcal{F}_6(x, y) = \operatorname{ran} x = \{\pi_2(z) \mid z \in x \wedge z \text{ is an ordered pair}\}$;
(vii) $\mathcal{F}_7(x, y) = \{\langle u, v, w\rangle \mid \langle u, v\rangle \in x, w \in y\}$;
(viii) $\mathcal{F}_8(x, y) = \{\langle u, w, v\rangle \mid \langle u, v\rangle \in x, w \in y\}$;
(ix) $\mathcal{F}_9(x, y) = \{\langle v, u\rangle \in y \times x \mid u = v\}$;
(x) $\mathcal{F}_{10}(x, y) = \{\langle v, u\rangle \in y \times x \mid u \in v\}$.

Proposition. The following can all be written as a finite combination of Gödel functions (i)–(vii).

$$\{x\}; \quad x \cup y; \quad x \cap y; \quad \langle x, y\rangle; \quad \langle x, y, z\rangle$$

Proposition. For every $i \in \{1, \dots, 10\}$, the statement $z = \mathcal{F}_i(x, y)$ can be written using a Δ_0 formula. Hence, these formulas are absolute.

Lemma (Gödel normal form)**.** For every Δ_0 formula $\varphi(x_1, \dots, x_n)$ with free variables contained in $\{x_1, \dots, x_n\}$, there is a term $\mathcal{F}_\varphi$ built from the symbols $\mathcal{F}_1, \dots, \mathcal{F}_{10}$ such that

$$\mathsf{ZF} \vdash \forall a_1, \dots, a_n.\, \mathcal{F}_\varphi(a_1, \dots, a_n) = \{\langle x_n, \dots, x_1\rangle \in a_n \times \dots \times a_1 \mid \varphi(x_1, \dots, x_n)\}$$

Remark. (i) The reversed order of the free variables is done purely for technical reasons.

(ii) $\mathcal{F}_2$ will correspond to disjunction for Δ_0 formulas, intersection will correspond to conjunction, $\mathcal{F}_3$ will give negation, and $\mathcal{F}_9$ and $\mathcal{F}_{10}$ will give atomic formulas.

(iii) $\mathcal{F}_7$ and $\mathcal{F}_8$ will deal with ordered n-tuples. For example, the triple $\langle x_1, x_2, x_3\rangle$ is formed using x_1 and $\langle x_2, x_3\rangle$, but it cannot be formed using $\langle x_1, x_2\rangle$ and x_3 without $\mathcal{F}_7$ or $\mathcal{F}_8$.

Proof. We show this by induction on the class Δ_0. We call a formula φ a *termed formula* if the conclusion of the lemma holds for φ; we aim to show that every Δ_0-formula is a termed formula. We will only use the logical symbols $\wedge, \vee, \neg, \exists$, and the only occurrence of existential quantification will be in formulas of the form

$$\varphi(x_1, \dots, x_n) \equiv \exists x_{n+1} \in x_j.\, \psi(x_1, \dots, x_{n+1})$$

where $j \leq m \leq n$. For example, we allow $\exists x_3 \in x_1.\,(x_1 \in x_2 \wedge x_3 = x_1)$, but we disallow $\exists x_1 \in x_2.\,\psi$ and $\exists x_3 \in x_1.\,(x_3 \in x_2 \wedge \exists x_4 \in x_1.\,\psi)$. Every Δ_0-formula is equivalent to one of this form. We allow for dummy variables, so $\varphi(x_1, x_2) \equiv x_1 \in x_2$ and $\varphi(x_1, x_2, x_3) \equiv x_1 \in x_2$ are distinct. This proof will take place in four parts: first some logical points, then we consider propositional formulas, then atomic formulas, and finally bounded existentials.

Part (i): logical points. We make the following remarks.

- If $\mathsf{ZF} \vdash \varphi(\mathbf{x}) \leftrightarrow \psi(\mathbf{x})$ and $\varphi(\mathbf{x})$ is a termed formula, then ψ is also a termed formula. This is immediate from the definition, since we can let $\mathcal{F}_\psi = \mathcal{F}_\varphi$.
- For all m, n, if $\varphi(x_1, \dots, x_n) \equiv \psi(x_1, \dots, x_m)$ and ψ is a termed formula, then so is φ. If $n \geq m$, we can show this by induction on n. The base case $n = m$ is trivial. For the inductive step, suppose

  $$\varphi(x_1, \dots, x_{n+1}) \equiv \psi(x_1, \dots, x_m)$$

  Then, we can write

  $$\varphi(x_1, \dots, x_{n+1}) \equiv \theta(x_1, \dots, x_n)$$

  where θ is a termed formula. Then

  $$\mathcal{F}_\varphi(a_1, \dots, a_n, a_{n+1}) = a_{n+1} \times \mathcal{F}_\theta(a_1, \dots, a_n) = \mathcal{F}_4(a_{n+1}, \mathcal{F}_\theta(a_1, \dots, a_n))$$

  giving the result by the inductive hypothesis. This is the reason for reversing the order: because the ordered triple $\langle x, y, z\rangle$ is $\langle x, \langle y, z\rangle\rangle$, the map

  $$\{\langle x_1, x_2\rangle \in a_1 \times a_2 \mid \theta(x_1, x_2)\} \mapsto \{\langle x_1, x_2, x_3\rangle \in a_1 \times a_2 \times a_3 \mid \theta(x_1, x_2)\}$$

  is much more complicated to implement in Gödel functions. We prove the case $n \leq m$ by induction; if

  $$\varphi(x_1, \dots, x_{n-1}) \equiv \psi(x_1, \dots, x_m)$$

  then

  $$\varphi(x_1, \dots, x_{n-1}) \equiv \theta(x_1, \dots, x_n)$$

and

$$\{0\} = \{\mathcal{F}_3(a_1, a_1)\} = \mathcal{F}_1(\mathcal{F}_3(a_1, a_1), \mathcal{F}_3(a_1, a_1))$$

Then

$$\begin{aligned}\mathcal{F}_\varphi(a_1, \dots, a_{n-1}) &= \{\langle x_{n-1}, \dots, x_1\rangle \in a_{n-1} \times \cdots \times a_1 \mid \varphi(x_1, \dots, x_{n-1})\} \\ &= \operatorname{ran}(\{\langle 0, x_{n-1}, \dots, x_1\rangle \in \{0\} \times a_{n-1} \times \cdots \times a_1 \mid \theta(x_1, \dots, x_{n-1}, 0)\}) \\ &= \mathcal{F}_6(\mathcal{F}_\theta(a_1, \dots, a_{n-1}, \mathcal{F}_1(\mathcal{F}_3(a_1, a_1), \mathcal{F}_3(a_1, a_1))), a_1)\end{aligned}$$

- If $\psi(x_1, \dots, x_n)$ is a termed formula and

  $$\varphi(x_1, \dots, x_{n+1}) = \psi(x_1, \dots, x_{n-1}, x_{n+1}/x_n)$$

  then φ is a termed formula. First, if $n = 1$, we have a termed formula $\psi(x_1)$ and consider $\psi(x_2/x_1)$. Then

  $$\begin{aligned}\mathcal{F}_\varphi(a_1, a_2) &= \{\langle x_2, x_1\rangle \in a_2 \times a_1 \mid \psi(x_2)\} \\ &= \left\{\langle x_2, x_1\rangle \mid x_1 \in a_1 \wedge x_2 \in \mathcal{F}_\psi(a_2)\right\} \\ &= \mathcal{F}_\psi(a_2) \times a_1 \\ &= \mathcal{F}_4(\mathcal{F}_\psi(a_2), a_1)\end{aligned}$$

  If $n > 1$, we have

  $$\begin{aligned}\mathcal{F}_\varphi(a_1, \dots, a_{n+1}) &= \left\{\langle x_{n+1}, \dots, x_1\rangle \mid x_n \in a_n \wedge \langle x_{n+1}, x_{n-1}, \dots, x_1\rangle \in \mathcal{F}_\psi(a_1, \dots, a_{n-1}, a_{n+1})\right\} \\ &= \mathcal{F}_8(\mathcal{F}_\psi(a_1, \dots, a_{n-1}, a_{n+1}), a_n)\end{aligned}$$

- If $\psi(x_1, x_2)$ is a termed formula, and

  $$\varphi(x_1, \dots, x_n) \equiv \psi(x_{n-1}/x_1, x_n/x_2)$$

  then φ is a termed formula. This is trivial if $n = 2$, so we assume $n > 2$. Then

  $$\begin{aligned}\mathcal{F}_\varphi(a_1, \dots, a_n) &= \left\{\langle x_n, \dots, x_1\rangle \in a_n \times \cdots \times a_1 \mid \langle x_n, x_{n-1}\rangle \in \mathcal{F}_\psi(a_{n-1}, a_n)\right\} \\ &= \mathcal{F}_7(\mathcal{F}_\psi(a_{n-1}, a_n), a_{n-2} \times \cdots \times a_1)\end{aligned}$$

Part (ii): propositional connectives.

- If φ is a termed formula, then so is $\neg\varphi$.

  $$\mathcal{F}_{\neg\varphi}(a_1, \dots, a_n) = (a_n \times \cdots \times a_1) \setminus \mathcal{F}_\varphi(a_1, \dots, a_n)$$

- If φ, ψ are termed formulas, then so is $\varphi \vee \psi$.

  $$\mathcal{F}_{\varphi\vee\psi}(a_1, \dots, a_n) = \mathcal{F}_\varphi(a_1, \dots, a_n) \cup \mathcal{F}_\psi(a_1, \dots, a_n)$$

  It is easy to see that unions can be formed using Gödel functions.

- Conjunctions are similar to disjunctions.

  $$\mathcal{F}_{\varphi\wedge\psi}(a_1, \dots, a_n) = \mathcal{F}_\varphi(a_1, \dots, a_n) \cap \mathcal{F}_\psi(a_1, \dots, a_n)$$

Part (iii): atomic formulas.

- Consider $\varphi(x_1,\dots,x_n) \equiv x_i = x_j$. We show that this is a termed formula for all $i, j \le n$. Suppose $i = 1$ and $j = 2$. In this case,

  $$\mathcal{F}_9(a_1,a_2) = \{\langle x_2,x_1\rangle \in a_2 \times a_1 \mid x_1 = x_2\}$$

  so $\mathcal{F}_\varphi$ is formed using $\mathcal{F}_9$ and the discussion on dummy variables. Now suppose $j \ge i$. We prove this by induction. First, if $i = j$, then

  $$\mathcal{F}_\varphi = \{\langle x_n,\dots,x_1\rangle \in a_n \times \cdots \times a_1 \mid x_i = x_i\} = a_n \times \cdots \times a_1$$

  Now, if $j = i + 1$, we let

  $$\theta(x_1,\dots,x_{i+1}) = (x_1 = x_2)[x_i/x_1, x_{i+1}/x_2]$$

  This is a termed formula by the result on substitutions. We thus obtain $\mathcal{F}_\varphi$ by adding the required dummy variables. Now suppose we have $\varphi(x_1,\dots,x_n) \equiv x_i = x_{j+1}$. Then we can write

  $$\varphi(x_1,\dots,x_{j+1}) = (x_i,x_j)[x_{j+1},x_j]$$

  which is a termed formula by substitution. This concludes the case $i \le j$ by induction. Finally, suppose $i > j$. As $x_i = x_j$ is logically equivalent to $x_j = x_i$, which is a termed formula, φ is also a termed formula.

- Now consider $\varphi(x_1,\dots,x_n) \equiv x_i \in x_j$. As with equality, we first consider the case $i = 1, j = 2$. In this case, we can form $\mathcal{F}_{10}$ with dummy variables. If $i = j$, the formula is always false, so we have

  $$\mathcal{F}_\varphi(a_1,\dots,a_n) = \varnothing = a_1 \setminus a_1 = \mathcal{F}_3(a_1,a_1)$$

  Now, let

  $$\psi(x_1,\dots,x_{n+2}) \equiv (x_i = x_{n+1}) \wedge (x_j = x_{n+2}) \wedge (x_{n+1} \in x_{n+2})$$

  We note that $x_{n+1} \in x_{n+2}$ is a termed formula as it is given by the substitution $(x_1 \in x_2)[x_{n+1}/x_1, x_{n+2}/x_2]$. The equalities are termed formulas as above, so ψ is a termed formula. Then

  $$\begin{aligned}\mathcal{F}_\varphi(a_1,\dots,a_n) = \operatorname{ran}\operatorname{ran}\{&\langle x_{n+2},\dots,x_1\rangle \times a_j \times a_i \times a_n \times \cdots \times a_1 \mid \\ &x_i = x_{n+1} \wedge x_j = x_{n+2} \wedge x_{n+1} \in x_{n+2}\} \\ = \mathcal{F}_6(&\mathcal{F}_6(\mathcal{F}_\psi(a_1,\dots,a_n),a_1),a_1)\end{aligned}$$

Part (iv): bounded quantifiers. We required that the only occurrence of $\exists$ was in the form

$$\varphi(x_1,\dots,x_n) \equiv \exists x_{m+1} \in x_j.\, \psi(x_1,\dots,x_{m+1})$$

where $j \le m \le n$. Due to this restriction, it suffices to show that if $\psi(x_1,\dots,x_{n+1})$ is a termed formula, then so is the formula

$$\varphi(x_1,\dots,x_n) \equiv \exists x_{n+1} \in x_j.\, \psi(x_1,\dots,x_{n+1})$$

Let $\theta(x_1,\dots,x_{n+1}) \equiv x_{n+1} \in x_j$. Then $\theta \wedge \psi$ is a termed formula. Now

$$\begin{aligned}\mathcal{F}_{\theta\wedge\psi}(a_1,\dots,a_n,\mathcal{F}_2(a_j,a_j)) &= \mathcal{F}_{\theta\wedge\psi}\Big(a_1,\dots,a_n,\bigcup a_j\Big) \\ &= \Big\{\langle x_{n+1},\dots,x_1\rangle \in \Big(\bigcup a_j\Big) \times a_n \times \cdots \times a_1 \mid \\ &\qquad x_{n+1} \in x_j \wedge \forall k \le n.\, x_k \in a_k \wedge \psi(x_1,\dots,x_{n+1})\Big\}\end{aligned}$$

So

$$\begin{aligned}\operatorname{ran}(\mathcal{F}_{\theta\wedge\psi}\big(a_1,\dots,a_n,\bigcup a_j\big)) &= \Big\{\langle x_n,\dots,x_1\rangle \in a_n \times \dots \times a_1 \mid \\ &\qquad \exists u.\, \langle u, x_n,\dots,x_1\rangle \in \mathcal{F}_{\theta\wedge\psi}\big(a_1,\dots,a_n,\bigcup a_j\big)\Big\} \\ &= \Big\{\langle x_n,\dots,x_1\rangle \in a_n \times \dots \times a_1 \mid \\ &\qquad \exists x_{n+1} \in x_j.\, \psi(x_1,\dots,x_{n+1})\Big\}\end{aligned}$$

□

Definition. A class C is *closed under Gödel functions* if whenever $x, y \in C$, we have $\mathcal{F}_i(x,y) \in C$ for $i \in \{1,\dots,10\}$. Given a set b, we let $\operatorname{cl}(b)$ be the smallest set C containing b as a subset that is closed under Gödel functions.

For example, $\operatorname{cl}(\varnothing) = \varnothing$, $a, b \in \operatorname{cl}(\{a,b\})$, and $\operatorname{cl}(b) = \operatorname{cl}(\operatorname{cl}(b))$.

Definition. Let b be a set. Define $\mathcal{D}^n(b)$ inductively by

$$\mathcal{D}^0(b) = b; \quad \mathcal{D}^{n+1}(b) = \{\mathcal{F}_i(x,y) \mid x, y \in \mathcal{D}^n(b), i \in \{1,\dots,10\}\}$$

One can easily check that $\operatorname{cl}(b) = \bigcup_{n\in\omega} \mathcal{D}^n(b)$.

Lemma. If M is a transitive class that is closed under Gödel functions, then M satisfies Δ_0-separation.

Proof. Let $\varphi(x_1,\dots,x_n)$ be a Δ_0-formula, and let $a, b_1,\dots,b_{i-1},b_{i+1},\dots,b_n \in M$. Let

$$Y = \{x_i \in a \mid \varphi(b_1,\dots,b_{i-1},x_i,b_{i+1},\dots,b_n)\}$$

We must show $Y \in M$. Let $\mathcal{F}_\varphi$ be the formula built from Gödel's normal form theorem. Then for any $c_1,\dots,c_n \in M$, we have

$$\mathcal{F}_\varphi(c_1,\dots,c_n) = \{\langle x_n,\dots,x_1\rangle \in c_n \times \dots \times c_1 \mid \varphi(x_1,\dots,x_n)\} \in M$$

Hence, as $\{b_j\} = \mathcal{F}_1(b_j,b_j) \in M$, we obtain

$$\mathcal{F}_\varphi(\{b_1\},\dots,\{b_{i-1}\},a,\{b_{i+1}\},\dots,\{b_n\}) \in M$$

Then, we can show that $Y \in M$ by taking the range $\mathcal{F}_6$ a total of $n - i$ times and then taking the domain $\mathcal{F}_5$. □

Theorem. For every transitive set M, the collection of definable subsets is

$$\operatorname{Def}(M) = \operatorname{cl}(M \cup \{M\}) \cap \mathcal{P}(M)$$

Proof. We first prove the forward direction. Let φ be a formula. Then φ^M is Δ_0, so there is a term $\mathcal{G}$ built from the Gödel functions $\mathcal{F}_1, \dots, \mathcal{F}_{10}$ such that for $a_1, \dots, a_n \in M$, we have

$$\{x \in M \mid (M, \in) \vDash \varphi(x, a_1, \dots, a_n)\} = \left\{x \in M \mid \varphi^M(x, a_1, \dots, a_n)\right\} = \mathcal{G}(M, a_1, \dots, a_n) \in \mathrm{cl}(M \cup \{M\})$$

We now show the converse. We first claim that if $\mathcal{G}$ is built from the Gödel functions, then for any $x, a_1, \dots, a_n$, the formulas

$$x = \mathcal{G}(a_1, \dots, a_n); \quad x \in \mathcal{G}(a_1, \dots, a_n)$$

are Δ_0. This can be proven inductively using the iterative construction of $\mathrm{cl}(M \cup \{M\})$. For example, if $X, Y \in \mathcal{D}^k(a_1, \dots, a_n)$, then $x = \mathcal{F}_1(X, Y)$ is equivalent to the statement

$$(\forall z \in x.\, z = X \vee z = Y) \wedge (\exists w \in x.\, w = X) \wedge (\exists w \in x.\, w = Y)$$

so the result holds for $\mathcal{F}_1$; very similar proofs show the result for both equality and membership for all other Gödel functions.

Let $Z \in \mathrm{cl}(M \cup \{M\}) \cap \mathcal{P}(M)$. Since $Z \in \mathrm{cl}(M \cup \{M\})$, we can fix a term $\mathcal{G}$ built from the $\mathcal{F}_1, \dots, \mathcal{F}_{10}$ such that $Z = \mathcal{G}(M, a_1, \dots, a_n)$. Let φ be a Δ_0 formula such that $x \in \mathcal{G}(M, a_1, \dots, a_n)$ if and only if $\varphi(x, M, a_1, \dots, a_n)$. Then $\mathcal{G}(M, a_1, \dots, a_n) = \{x \in M \mid \varphi(x, M, a_1, \dots, a_n)\}$ as $Z \subseteq M$. It therefore remains to prove that there is a formula ψ such that

$$\psi^M(x, a_1, \dots, a_n) \leftrightarrow \varphi(x, M, a_1, \dots, a_n)$$

For example, we can define ψ from φ by the following replacements.

(i) $\exists v_i \in M \mapsto \exists v_i$;

(ii) $v_i \in M \mapsto v_i = v_i$;

(iii) $M = M \mapsto v_0 = v_0$;

(iv) $M \in M, M \in v_i, M = v_i \mapsto v_0 \neq v_0$.

Finally, we obtain

$$Z = \mathcal{G}(M, a_1, \dots, a_n) = \left\{x \in M \mid \psi^M(x, a_1, \dots, a_n)\right\} \in \mathrm{Def}(M)$$

□

2.4 The axiom of constructibility

Definition. The *axiom of constructibility* is the statement $\mathrm{V} = \mathrm{L}$. Equivalently, $\forall x.\, \exists \alpha \in \mathrm{Ord}.\, (x \in \mathrm{L}_\alpha)$.

We will show that if ZF is consistent, then so is ZF + (V = L), by demonstrating that L is a model of ZF + (V = L). To do this, we will show that being constructible is absolute.

Lemma. $Z = \mathrm{cl}(M)$ is Δ_1^{ZF}.

Proof. The Π_1 definition is simply being the smallest set closed under Gödel functions. More explicitly,

$$\forall W. \left(M \cup \{M\} \subseteq W \wedge \forall x, y \in W. \bigwedge_{i \leq 10} \mathcal{F}_i(x, y) \in W \right) \to Z \subseteq W$$

The Σ_1 definition will use the inductive definition of the closure.

$$\begin{aligned} \exists W. W \text{ is a function} \wedge \operatorname{dom} W = \omega \wedge Z = \bigcup \operatorname{ran} W \\ \wedge W(0) = M \wedge W(n) \subseteq W(n+1) \\ \wedge \left(\forall x, y \in W(n). \bigwedge_{i \leq 10} \mathcal{F}_i(x, y) \in W(n+1) \right) \\ \wedge \left(\forall z \in W(n+1). \exists x, y \in W(n). \bigvee_{i \leq 10} z = \mathcal{F}_i(x, y) \right) \end{aligned}$$

□

Lemma. The function mapping $\alpha \mapsto \mathrm{L}_\alpha$ is absolute between transitive models of ZF.

Proof. Define $G : \mathrm{Ord} \times \mathrm{V} \to \mathrm{V}$ by

$$G(\alpha, x) = \begin{cases} \mathrm{cl}(x(\beta) \cup \{x(\beta)\}) & \text{if } \alpha = \beta + 1 \text{ and } x \text{ is a function with domain } \beta \\ \bigcup_{\beta < \alpha} x(\beta) & \text{if } \alpha \text{ is a limit} \\ \varnothing & \text{otherwise} \end{cases}$$

All of these conditions and constructions are absolute, so G is an absolute function. Therefore, by transfinite recursion, there exists $F : \mathrm{Ord} \to \mathrm{V}$ where $F : \alpha \mapsto G(x, F|_\alpha)$. By absoluteness of transfinite recursion, F is absolute. Finally, $F(\alpha) = \mathrm{L}_\alpha$ for all ordinal α. □

Theorem. (i) L satisfies the axiom of constructibility.
(ii) L is the smallest inner model of ZF. That is, if M is an inner model of ZF, then $\mathrm{L} \subseteq M$.

Proof. Part (i). We must show

$$(\forall x. \exists \alpha \in \mathrm{Ord}. x \in \mathrm{L}_\alpha)^{\mathrm{L}}$$

which is

$$\forall x \in \mathrm{L}. \exists \alpha \in \mathrm{Ord}. x \in (\mathrm{L}_\alpha)^{\mathrm{L}}$$

Since the L_α hierarchy is absolute, $x \in (\mathrm{L}_\alpha)^{\mathrm{L}}$ if and only if $x \in \mathrm{L}_\alpha$. As L contains every ordinal, if $x \in \mathrm{L}$ then $x \in \mathrm{L}_\alpha$ for some α, and thus $x \in (\mathrm{L}_\alpha)^{\mathrm{L}}$. Hence $\mathrm{L} \vDash \alpha \in \mathrm{L} \wedge x \in \mathrm{L}_\alpha$.

Part (ii). Let M be an arbitrary inner model of ZF. We construct L inside M to give L^M. By absoluteness, for every $\alpha \in M \cap \mathrm{Ord}$, we have $\mathrm{L}_\alpha = (\mathrm{L}_\alpha)^M$. Thus $\mathrm{L}_\alpha \subseteq M$ for every $\alpha \in M \cap \mathrm{Ord} = \mathrm{Ord}$. Hence $\mathrm{L} \subseteq M$ as required. □

2.5 Well-ordering the universe

We will show that L satisfies a strong version of the axiom of choice, namely that there is a definable global well-order. We will define well-orderings $<_\alpha$ on L_α such that $<_{\alpha+1}$ *end-extends* $<_\alpha$: if $y \in \mathrm{L}_\alpha$ and $x \in \mathrm{L}_{\alpha+1} \setminus \mathrm{L}_\alpha$, then $y <_{\alpha+1} x$. Then we set $<_\mathrm{L} = \bigcup_\alpha <_\alpha$.

Theorem. There is a well-ordering of L.

Proof. For each ordinal α, we will construct a well-order $<_\alpha$ on L_α such that if $\alpha < \beta$, the following hold:

(i) if $x <_\alpha y$ then $x <_\beta y$; and

(ii) if $x \in \mathrm{L}_\alpha$ and $y \in \mathrm{L}_\beta \setminus \mathrm{L}_\alpha$, then $x <_\beta y$.

For limit cases, we take unions:

$$<_\gamma = \bigcup_{\alpha<\gamma} <_\gamma$$

We now describe the construction of $<_{\alpha+1}$. To do this, we consider the ordering on L_α, and append the singleton $\{\mathrm{L}_\alpha\}$. We then follow that by the elements of $\mathcal{D}(\mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\}) \setminus (\mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\})$. We then add $\mathcal{D}^2(\mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\}) \setminus \mathcal{D}(\mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\})$, and so forth. In order to do this, we define $<^n_{\alpha+1}$ for $n \in \omega$ as follows.

(i) $<^0_{\alpha+1}$ is the well-ordering of $\mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\}$ given by making $\{\mathrm{L}_\alpha\}$ the maximal element.

(ii) Suppose that $<^n_{\alpha+1}$ is defined. We end-extend $<^n_{\alpha+1}$ to form $<^{n+1}_{\alpha+1}$ as follows. Suppose $x, y \notin \mathcal{D}^n(\mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\})$. We say $x <^{n+1}_{\alpha+1} y$ if either

  (a) the least $i \leq 10$ such that $\exists u, v \in \mathcal{D}^n(\mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\})$ with $x = \mathcal{F}_i(u, v)$ is less than the least $i \leq 10$ such that $\exists u, v \in \mathcal{D}^n(\mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\})$ with $y = \mathcal{F}_i(u, v)$; or

  (b) these indices i are equal, and the $<^n_{\alpha+1}$-least $u \in \mathcal{D}^n(\mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\})$ such that there exists $v \in \mathcal{D}^n(\mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\})$ with $x = \mathcal{F}_i(u, v)$ is less than the $<^n_{\alpha+1}$-least $u \in \mathcal{D}^n(\mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\})$ such that there exists $v \in \mathcal{D}^n(\mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\})$ with $y = \mathcal{F}_i(u, v)$; or

  (c) both of these coincide, and $<^n_{\alpha+1}$-least $v \in \mathcal{D}^n(\mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\})$ with $x = \mathcal{F}_i(u, v)$ is less than the least $v \in \mathcal{D}^n(\mathrm{L}_\alpha \cup \{\mathrm{L}_\alpha\})$ with $y = \mathcal{F}_i(u, v)$.

□

The restriction of $<_\mathrm{L}$ to any set $x \in \mathrm{L}$ is a well-ordering of x. Since every set can be well-ordered, the axiom of choice holds.

Lemma. The relation $<_\mathrm{L}$ is Σ_1-definable. Moreover, for every limit ordinal δ and $y \in \mathrm{L}_\delta$, we have $x <_\mathrm{L} y$ if and only if $x \in \mathrm{L}_\delta$ and $(\mathrm{L}_\delta, \in) \vDash x <_\mathrm{L} y$.

2.6 The generalised continuum hypothesis in L

Lemma. (ZFC)

(i) For all $n \in \omega$, we have $\mathrm{L}_n = \mathrm{V}_n$.

(ii) If M is infinite, then $|M| = |\mathrm{Def}(M)|$.

(iii) If α is an infinite ordinal, then $|\mathrm{L}_\alpha| = |\alpha|$.

Lemma (Gödel's condensation lemma)**.** For every limit ordinal δ, if $(M, \in) \prec (L_\delta, \in)$, then there exists some $\beta \leq \delta$ such that $(M, \in) \cong (\mathrm{L}_\beta, \in)$.

Proof. Let $\pi : (M, \in) \to (N, \in)$ be the Mostowski collapse, and set $\beta = N \cap \mathrm{Ord}$. Since N is transitive, $\beta \in \mathrm{Ord}$. We will prove that $\beta \leq \delta$ and $N = \mathrm{L}_\beta$.

First, suppose $\delta < \beta$. Then $\delta \in N$, so $\pi^{-1}(\delta) \in M$. Since being an ordinal is absolute between transitive models, $N \vDash \delta \in \mathrm{Ord}$, so $M \vDash \pi^{-1}(\delta) \in \mathrm{Ord}$. Note that this does not immediately imply that $\pi^{-1}(\delta)$ is an ordinal in V since M is not necessarily transitive. But as $M \prec \mathrm{L}_\delta$, we obtain $\mathrm{L}_\delta \vDash \pi^{-1}(\delta) \in \mathrm{Ord}$, and since L_δ is transitive, $\pi^{-1}(\delta)$ is an ordinal in V.

Also, $M \vDash x \in \pi^{-1}(\delta)$ if and only if $N \vDash \pi(x) \in \delta$. Hence,

$$\pi : (\pi^{-1}(\delta) \cap M) \to \delta$$

is an isomorphism. Therefore, the order type of $\pi^{-1}(\delta) \cap M$ is δ. Let $f : \delta \to \pi^{-1}(\delta) \cap M$ be a strictly increasing enumeration. Then, for any $\alpha \in \delta$, we must have $\alpha \leq f(\alpha) < \pi^{-1}(\delta)$. Hence $\delta \leq \pi^{-1}(\delta)$. On the other hand, $\pi^{-1}(\delta) \in M \prec \mathrm{L}_\delta$, so $\pi^{-1}(\delta) < \delta$. This gives a contradiction.

We now show $\beta > 0$. Since

$$\mathrm{L}_\delta \vDash \exists x. \forall y \in x. (y \neq y)$$

the elementary substructure M must also believe this statement, and so N does. In particular, since N believes in the existence of an empty set, we must have $\varnothing \in N \cap \mathrm{Ord} = \beta$ as required.

We show β is a limit. We know that

$$\mathrm{L}_\delta \vDash \forall \alpha \in \mathrm{Ord}. \exists x. x = \alpha + 1$$

So M and hence N believe this statement. Let $\alpha \in \beta = N \cap \mathrm{Ord}$, then by absoluteness, $\alpha + 1 \in N$.

Now we show $\mathrm{L}_\beta \subseteq N$.

$$\mathrm{L}_\delta \vDash \forall \alpha \in \mathrm{Ord}. \exists y. y = \mathrm{L}_\alpha$$

So N satisfies this sentence. Since the L_α hierarchy is absolute, for all $\alpha \in N \cap \mathrm{Ord} = \beta$, we have $\mathrm{L}_\alpha \in N$.

Finally, we show $N \subseteq \mathrm{L}_\beta$.

$$\mathrm{L}_\delta \vDash \forall x. \exists y. \exists z. y \in \mathrm{Ord} \wedge z = \mathrm{L}_y \wedge x \in z$$

As N satisfies this sentence, for a fixed $a \in N$ there are $\gamma \in N$ and $z \in N$ such that

$$N \vDash \gamma \in \mathrm{Ord} \wedge z = \mathrm{L}_\gamma \wedge a \in z$$

By absoluteness, $a \in \mathrm{L}_\gamma \subseteq \mathrm{L}_\beta$ as required. $\square$

Theorem. If $\mathrm{V} = \mathrm{L}$, then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α. In particular, GCH holds.

Proof. We will show that $\mathcal{P}(\omega_\alpha) \subseteq \mathrm{L}_{\omega_{\alpha+1}}$. Then, as $\left|\mathrm{L}_{\omega_{\alpha+1}}\right| = \aleph_{\alpha+1}$, the proof follows. To do this, it suffices to show that if $X \subseteq \omega_\alpha$, then there exists some $\gamma < \omega_{\alpha+1}$ such that $X \in \mathrm{L}_\gamma$.

Let $X \subseteq \omega_\alpha$ and let $\delta > \omega_\alpha$ be a limit ordinal such that $X \in \mathrm{L}_\delta$. Let M be an elementary submodel of L_δ such that $\omega_\alpha \subseteq M$, $X \in M$, and $|M| = \aleph_\alpha$. This exists by the downward Löwenheim–Skolem theorem. By Gödel's condensation lemma, if N is the Mostowski collapse of M, then there is a limit ordinal $\gamma \leq \delta$ such that $N = \mathrm{L}_\gamma$. As $|N| = |M| = \aleph_\alpha$, we have $\left|\mathrm{L}_\gamma\right| = \aleph_\alpha$, so $\gamma < \omega_{\alpha+1}$. Finally, as $\omega_\alpha \subseteq M$, the collapsing map is the identity on ω_α. Thus, the map fixes X, and so $X \in \mathrm{L}_\gamma$. □

This gives the following theorem.

Theorem. Con(ZF) implies Con(ZFC + V = L + GCH).

Proof. We have shown that there is a definable class L such that ZF proves

$$(\mathsf{ZFC} + \mathsf{V} = \mathsf{L} + \mathsf{GCH})^{\mathrm{L}}$$

Suppose that ZFC + V = L + GCH were inconsistent. Then fix φ such that

$$\mathsf{ZFC} + \mathsf{V} = \mathsf{L} + \mathsf{GCH} \vdash \varphi \wedge \neg\varphi$$

Then

$$\mathsf{ZF} \vdash (\varphi \wedge \neg\varphi)^L$$

By relativisation, $\varphi^L \wedge \neg(\varphi^L)$. Hence ZF is inconsistent. □

Lemma (Shepherdson)**.** There is no class W such that

$$\mathsf{ZFC} \vdash W \text{ is an inner model} \wedge (\neg\mathsf{CH})^W$$

Therefore, the technique of inner models does not let us prove the independence of CH from ZFC. In order to do this, we will introduce the notion of *forcing*.

2.7 Combinatorial properties

Definition. Let Ω be either a regular cardinal or the class of all ordinals. A subclass $C \subseteq \Omega$ is said to be a *club*, or *closed and unbounded*, if it is
(i) *closed*: for all $\gamma \in \Omega$, we have $\sup(C \cap \gamma) \in C$;
(ii) *unbounded*: for all $\alpha \in \Omega$ there exists $\beta \in C$ with $\beta > \alpha$.
A class $S \subseteq \Omega$ is *stationary* if it intersects every club.

Note that being a stationary class for Ord is not first-order definable.

The property $\diamond$ states that there is a single sequence of length ω_1 which can approximate any subset of ω_1 in a suitable sense.

Definition. We say that the *diamond principle* $\diamond$ holds if there is a sequence $(A_\alpha)_{\alpha<\omega_1}$ such that

(i) for each $\alpha < \omega_1$, we have $A_\alpha \subseteq \alpha$; and
(ii) for all $X \subseteq \omega_1$, the set $\{\alpha \mid X \cap \alpha = A_\alpha\}$ is stationary.

Lemma. $\mathsf{ZF} \vdash \diamondsuit \to \mathsf{CH}$.

Proof. If $(A_\alpha)_{\alpha<\omega_1}$ is a $\diamondsuit$-sequence, then for all $X \subseteq \omega$, there is $\alpha > \omega$ such that $X = A_\alpha$. Thus $\{A_\alpha \mid \alpha \in \omega_1 \wedge A_\alpha \subseteq \omega\} = \mathcal{P}(\omega)$. □

Theorem. If V = L, then $\diamondsuit$ holds.

Remark. $\diamondsuit$ is used in many inductive constructions in L to build combinatorial objects such as Suslin trees.

Definition. Let κ be an uncountable cardinal. Then the *square principle* $\square_\kappa$ is the assertion that there exists a sequence (C_α) indexed by the limit ordinals α in κ^+, such that
(i) C_α is a club subset of α;
(ii) if β is a limit ordinal of C_α then $C_\beta = C_\alpha \cap \beta$; and
(iii) if $\mathrm{cf}(\alpha) < \kappa$ then $|C_\alpha| < \kappa$.

Theorem (Jensen)**.** If V = L, then $\square_\kappa$ holds for every uncountable cardinal κ.

Lemma. If $\square_{\omega_1}$, then there exists a stationary set $S \subseteq \{\beta \in \omega_2 \mid \mathrm{cf}(\beta) = \omega\}$ such that for all $\alpha \in \omega_2$ with $\mathrm{cf}(\alpha) = \omega_1$, $S \cap \alpha$ is not stationary in α.

Remark. If κ is a weakly compact cardinal, then every stationary subset of κ *reflects*: there is $\alpha \in \kappa$ such that $S \cap \alpha$ is stationary in α. In fact, the claim that every stationary subset of $\{\beta \in \omega_2 \mid \mathrm{cf}(\beta) = \omega\}$ reflects at a point of cofinality ω_1 is equiconsistent with ZFC together with the assertion that there is a Mahlo cardinal.

3 Forcing

3.1 Introduction

The idea behind forcing is to widen a given model of ZFC to 'add lots of reals'. But if we work over V, we already have added all of the sets, so there is nothing left to add. Instead, we will work over countable transitive set models of ZFC. However, this means that we will not immediately get $\mathsf{Con}(\mathsf{ZF}) \to \mathsf{Con}(\mathsf{ZFC} + \neg\mathsf{CH})$. We will then use the reflection theorem to obtain this result.

If M is such a countable transitive model, we want to add ω_2^M-many reals to M. We will try to do this in a 'minimal way'; for example, we do not want to add any ordinals. This gives us much more control over the model that we build.

Recall the argument that the sentence $\varphi(x) \equiv \exists x.\, x^2 = 2$ is independent of the axioms of fields: we began with a field in which the sentence failed, namely $\mathbb{Q}$, and then extended it in a minimal way to $\mathbb{Q}\left[\sqrt{2}\right]$. The model $\mathbb{Q}\left[\sqrt{2}\right]$ does not just contain $\mathbb{Q} \cup \left\{\sqrt{2}\right\}$, it also contains everything that can be built from $\mathbb{Q}$ and $\sqrt{2}$ using the axioms of fields. The field $\mathbb{Q}\left[\sqrt{2}\right]$ is the minimal field extension of $\mathbb{Q}$ satisfying φ.

We may encounter some difficulties when adding arbitrary reals to our model. Suppose that M is of the form L_γ, where γ is a countable ordinal. Then γ can be coded as a subset c of ω, which can be viewed as a real. If we added c to M, we could decode it to form $\gamma = \mathrm{Ord} \cap M$. This would violate the principle of not adding any new ordinals.

Suppose we enumerate all formulas as $\{\varphi_n \mid n \in \omega\}$. Let $r = \{n \mid M \vDash \varphi_n\}$. If we added r to M, we could then build a truth predicate for M. This would cause self-referential problems discussed by Tarski.

The main issues we must overcome are the following.

(i) We need a method to choose the ω_2^M-many subsets of M to be added.

(ii) Given these, we need to ensure that the extension satisfies ZFC.

(iii) We must ensure that ω_1^M and ω_2^M are still cardinals in the extension.

We will build these reals from within M itself. Note that if r is a real, then each of its finite decimal approximations is already in M. The issue is that from within M, we do not know what the real we want to add is. So we may not know from within M which reals we will add. Instead, we will add a *generic* real. To be generic, we will not specify any particular digits, but its decimal expansion will contain every finite sequence. We will call a specification *dense* if any finite approximation can be extended to one satisfying the specification. For example, 'beginning with a 7' is not dense, but 'containing the subsequence 746' is dense. We will define that a real is generic precisely when it meets every dense specification.

Note that there are explicit, absolute bijections $f : \mathcal{P}(\omega) \to \omega^\omega$, $g : \omega^\omega \to 2^\omega$, $h : 2^\omega \to \mathbb{R}$ and so on. So if $M \vDash$ ZFC, knowledge of $\mathcal{P}^M(\omega)$ gives us $(\omega^\omega)^M, (2^\omega)^M, \mathbb{R}^M$. Because of this, by a 'real' we mean either an element of $\mathbb{R}$, a function $\omega \to \omega$, a function $\omega \to 2$, or a subset of ω. In formal arguments, reals will normally be either subsets of ω or functions $\omega \to 2$.

The axiom of choice is not needed in the basic machinery of forcing, so we will work primarily over ZF and state explicitly where choice is used.

3.2 Forcing posets

Definition. A *preorder* is a pair $(\mathbb{P}, \leq)$ such that
- $\mathbb{P}$ is nonempty;
- $\leq$ is a binary relation on $\mathbb{P}$;
- $\leq$ is transitive, so $p \leq q$ and $q \leq r$ implies $p \leq r$;
- $\leq$ is reflexive, so $p \leq p$.

A preorder is called a *partial order* if $\leq$ is antisymmetric, so $p \leq q$ and $q \leq p$ implies $p = q$.

Definition. A *forcing poset* is a triple $(\mathbb{P}, \leq_\mathbb{P}, \mathbb{1}_\mathbb{P})$, where $(\mathbb{P}, \leq_\mathbb{P})$ is a preorder and $\mathbb{1}_\mathbb{P}$ is a maximal element. Elements of $\mathbb{P}$ are called *conditions*, and we say q is *stronger* than p or an *extension* of p if $q \leq p$. We say that p, q are *compatible*, written $p \parallel_\mathbb{P} q$, if there exists r such that $r \leq_{\mathbb{P}} p, q$. Otherwise, we say they are *incompatible*, written $p \perp q$.

Remark. In some texts, the partial order is reversed. This is called *Jerusalem notation*.

The notation $\mathbb{P} \in M$ abbreviates $(\mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}}) \in M$. Note that by transitivity if $\mathbb{P}$ is an element of M, then $\mathbb{1}_{\mathbb{P}} \in M$, but we do not necessarily have $\leq_{\mathbb{P}} \in M$.

Definition. A preorder is *separative* if whenever $p \neq q$, exactly one of the following two cases holds:
(i) $q \leq p$ and $p \not\leq q$; or
(ii) there exists $r \leq q$ such that $r \perp p$.

Proposition. (i) If $(\mathbb{P}, \leq)$ is a separative preorder, it is a partial order.
(ii) If $(\mathbb{P}, \leq)$ is a poset, then it is separative if and only if whenever $q \not\leq p$, there is $r \leq q$ such that $r \perp p$.

Proposition. Suppose that $(\mathbb{P}, \leq)$ is a preorder. Define $p \sim q$ by

$$p \sim q \leftrightarrow \forall r \in P. (r \parallel p \leftrightarrow r \parallel q)$$

Then there is a separative preorder on ${}^{\mathbb{P}}\!/_{\sim}$ such that

$$[p] \perp [q] \leftrightarrow p \perp q$$

and if $\mathbb{P}$ has a maximal element, so does ${}^{\mathbb{P}}\!/_{\sim}$.

Example. For sets I, J, we let $\mathrm{Fn}(I, J)$ denote the set of all finite partial functions from I to J.

$$\mathrm{Fn}(I, J) = \{p \mid |p| < \omega \wedge p \text{ is a function} \wedge \operatorname{dom} p \subseteq I \wedge \operatorname{ran} p \subseteq J\}$$

We let $\leq$ be the reverse inclusion on $\mathrm{Fn}(I, J)$, so $q \leq p$ if and only if $q \supseteq p$. The maximal element $\mathbb{1}$ is the empty set. Then $(\mathrm{Fn}(I, J), \supseteq, \varnothing)$ is a forcing poset, and moreover, the preorder is separative.

Remark. When α is an ordinal, the forcing poset $\mathrm{Fn}(\alpha \times \omega, 2)$ is often written $\mathrm{Add}(\omega, \alpha)$, denoting the idea that we are adding α-many subsets of ω.

3.3 Chains and Δ-systems

Definition. Let $\mathbb{P}$ be a forcing poset.
(i) A *chain* is a subset $C \subseteq \mathbb{P}$ such that for every $p, q \in C$, either $p \leq q$ or $q \leq p$.
(ii) An *antichain* is a subset $A \subseteq \mathbb{P}$ such that for every $p, q \in A$, either $p = q$ or $p \perp q$. An antichain is *maximal* if it is not strictly contained in any other antichain.
(iii) We say that $\mathbb{P}$ has the *countable chain condition* if every antichain is countable.

Example. (i) Consider the tree $\mathrm{Fn}(\omega, 2)$. A chain is a branch through the tree, and an antichain is a collection of points on different branches.

(ii) The set of functions $\{\{\langle 0,0\rangle, \langle 1,n\rangle\} \mid n \in \omega\}$ forms an antichain of length ω in $\mathrm{Fn}(I,\omega)$ if $\{0,1\} \subseteq I$.

Definition. A family of sets $\mathcal{A}$ forms a Δ-*system with root* R when $X \cap Y = R$ for all $X \neq Y$ in $\mathcal{A}$.

Example. If $R = \varnothing$, then $\mathcal{A}$ is a family of pairwise disjoint sets.

Definition. Let A be a set and θ a cardinal. Then we write $[A]^\theta$ for the set of subsets of A of size θ.

$$[A]^\theta = \{x \subseteq A \mid |x| = \theta\}$$

We write $[A]^{<\theta}$ for the set of subsets of A of size strictly less than θ.

$$[A]^{<\theta} = \{x \subseteq A \mid |x| < \theta\}$$

Similarly, $[A]^{\leq\theta} = [A]^\theta \cup [A]^{<\theta}$.

Recall that for regular cardinals κ, if $\mathcal{F}$ is a family of sets of size less than κ and each element of $\mathcal{F}$ has size less than κ, then $\bigcup \mathcal{F}$ has size less than κ.

Lemma (Δ-system lemma). (ZFC) Let κ be an uncountable regular cardinal, and let $\mathcal{A}$ be a family of finite sets with $|\mathcal{A}| = \kappa$. Then there exists $\mathcal{B} \in [\mathcal{A}]^\kappa$ that forms a Δ-system.

Proof. To begin, we construct $\mathcal{C} \in [\mathcal{A}]^\kappa$ such that all elements of $\mathcal{C}$ have the same cardinality. By assumption, each element of $\mathcal{A}$ is finite, and so we can define $Y_n = \{X \in \mathcal{A} \mid |X| = n\}$, and suppose each of the Y_n had size less than κ. Then $|\mathcal{A}| = |\bigcup Y_n| < \kappa$, giving a contradiction.

Fix $n \in \omega$ such that $\mathcal{C} = Y_n$ has size κ. We show by induction on n that if $\mathcal{C} = \{X \in \mathcal{A} \mid |X| = n\}$, then there is $\mathcal{B} \subseteq \mathcal{C}$ of size κ that forms a Δ-system. If $n = 1$, we have a collection of pairwise disjoint singletons, so $\mathcal{C}$ is already a Δ-system with root $\varnothing$ as required. Now suppose $n > 1$ and the claim holds for $n-1$. For each $p \in \bigcup \mathcal{C}$, let $C_p = \{X \in \mathcal{C} \mid p \in X\}$. There are two cases to consider.

Suppose $\left|C_p\right| = \kappa$ for some $p \in \bigcup \mathcal{C}$. Then for such a p, we set $\mathcal{D} = \left\{X \setminus \{p\} \mid X \in C_p\right\}$. This set has size κ, and each element of $\mathcal{D}$ has size $n-1$. By the inductive hypothesis, we can find some $\mathcal{E} \in [\mathcal{D}]^\kappa$ such that $\mathcal{E}$ forms a Δ-system with root R. Then $\{Y \cup \{b\} \mid Y \in \mathcal{E}\}$ is a Δ-system with root $R \cup \{p\}$.

Now suppose all of the C_p have size less than κ. Then as κ is regular, for any set S of size less than κ,

$$\{X \in \mathcal{C} \mid X \cap S \neq \varnothing\} = \bigcup_{p \in S} C_p$$

has size less than κ. Therefore, there exists some $X \in \mathcal{C}$ such that $X \cap S = \varnothing$. We recursively choose $X_\alpha \in \mathcal{C}$ for each $\alpha < \kappa$ such that $X_\alpha \cap \bigcup_{\beta<\alpha} X_\beta = \varnothing$. Then $\{X_\alpha \mid \alpha < \kappa\} \in [\mathcal{C}]^\kappa$ is a Δ-system with empty root. $\square$

We can show that assumptions in the above lemma were required.

Proposition. Suppose κ is ω or singular. Then there exists a family $\mathcal{A}$ of finite sets with $|\mathcal{A}| = \kappa$ but no $\mathcal{B} \in [\mathcal{A}]^\kappa$ forms a Δ-system.

Lemma. (ZFC) $\mathrm{Fn}(I, J)$ has the countable chain condition if and only if I is empty or J is countable.

Proof. First, we observe that if I or J are empty, then $\mathrm{Fn}(I, J)$ is empty and so trivially has the countable chain condition. Now let us assume that both I and J are nonempty.

Suppose that J is uncountable. Then for any $i \in I$, the set

$$\{\{\langle i, j\rangle\} \mid j \in J\}$$

is an uncountable antichain.

Now suppose J is countable, and let $\{p_\alpha \mid \alpha \in \omega_1\}$ be a collection of distinct elements of $\mathrm{Fn}(I, J)$. Let $\mathcal{A} = \{\mathrm{dom}\, p_\alpha \mid \alpha \in \omega_1\}$, which is a collection of ω_1-many finite sets. By the Δ-system lemma, we can find an uncountable subset $\mathcal{B} \subseteq \mathcal{A}$ with a root $R \subseteq I$. By definition, $R \subseteq \mathrm{dom}(p_\alpha)$ for all $\mathrm{dom}\, p_\alpha \in \mathcal{B}$, the root R must be finite. Since J is countable, there are only countably many functions $R \to J$. Therefore, as $\mathcal{B}$ is uncountable, there are $\alpha \neq \beta$ such that $\mathrm{dom}\, p_\alpha$ and $\mathrm{dom}\, p_\beta$ are both in $\mathcal{B}$ and $p_\alpha|_R = p_\beta|_R$. But then since R is a root, $\mathrm{dom}\, p_\alpha \cap \mathrm{dom}\, p_\beta = R$, so $p_\alpha \parallel p_\beta$, witnessed by their union $p_\alpha \cup p_\beta$. So the $\{p_\alpha \mid \alpha \in \omega_1\}$ cannot form an antichain. □

3.4 Dense sets and genericity

Definition. Let $\mathbb{P}$ be a forcing poset.
(i) $D \subseteq \mathbb{P}$ is *dense* if for all $p \in \mathbb{P}$ there exists $q \in D$ such that $q \leq p$.
(ii) $D \subseteq \mathbb{P}$ is *open* if for all $p \in D$ and $q \in \mathbb{P}$, if $q \leq p$ then $q \in D$.

A set of conditions is dense if every condition can be extended to one in that set, and a set is open if it is closed under strengthening conditions.

Example. Let I be infinite and J nonempty. Then for all $i \in I$ and $j \in J$, the following are dense.

(i) $D_i = \{q \in \mathrm{Fn}(I, J) \mid i \in \mathrm{dom}\, q\}$;

(ii) $R_j = \{q \in \mathrm{Fn}(I, J) \mid j \in \mathrm{ran}\, q\}$.

Definition. A subset G of a forcing poset $\mathbb{P}$ is a *filter* if
(i) $\mathbb{1} \in G$;
(ii) for all $p, q \in G$ there is $r \in G$ such that $r \leq p$ and $r \leq q$;
(iii) for all $p, q \in G$, if $q \leq p$ and $q \in G$ then $p \in G$.
A filter G is $\mathbb{P}$-*generic over* M if $G \cap D$ is nonempty for every $\mathbb{P}$-dense subset $D \in M$.

Lemma (generic filter existence lemma)**.** Let M be an arbitrary countable set, and let $\mathbb{P} \in M$ be a forcing poset. Then for any condition $p \in \mathbb{P}$, there is a filter $G \subseteq \mathbb{P}$ containing p which is $\mathbb{P}$-generic over M.

Proof. Let $(D_n)_{n\in\omega}$ enumerate all dense subsets of $\mathbb{P}$ which lie in M. We inductively define $X \subseteq \mathbb{P}$ by $X = \{q_n \mid n \in \omega\}$ as follows. Let $q_0 = p$, and given q_n, we choose $q_{n+1} \in D_n$ such that $q_{n+1} \leq q_n$. Finally, let $G = \{r \in \mathbb{P} \mid \exists n.\, q_n \leq r\}$. Then G is a filter as the q_n form a chain, and it is clearly generic. $\square$

Definition. A condition $p \in \mathbb{P}$ is *minimal* if whenever $q \leq p$, we have $q = p$.

Lemma. Let M be a countable transitive model of ZF, and let $\mathbb{P} \in M$ be a separative partial order. Then either $\mathbb{P}$ has a minimal element, or for every filter G which is $\mathbb{P}$-generic over M, we have $G \notin M$.

Proof. Suppose $\mathbb{P}$ has no minimal element. Let G be a $\mathbb{P}$-generic filter over M. We show that if $F \subseteq \mathbb{P}$ is a filter in M, then the set $D_F = \mathbb{P} \setminus F \in M$ is a dense set. Then $G \cap D_F$ is nonempty for all filters F, so G cannot be equal to any filter $F \in M$.

Fix $p \in \mathbb{P}$. If $p \notin F$, then $p \in D_F$ as required. Otherwise, suppose $p \in F$. As p is not minimal, we can fix some $q \in F$ with $q < p$. Then $p \nleq q$, so by separativity, there is $r \leq p$ such that $r \perp q$. But all conditions in F are compatible, so one of r and q is not in F. $\square$

Proposition. For sets I, J such that $|I| \geq \omega$ and $|J| \geq 2$, the forcing poset $\mathrm{Fn}(I, J)$ is a separative partial order without a minimal element.

Proposition. (ZFC) Let $\mathbb{P} \in M$ be a forcing poset, and let $G \subseteq \mathbb{P}$. Then the following are equivalent.

(i) G is $\mathbb{P}$-generic over M, that is, for all dense sets $D \in M$, we have $G \cap D \neq \varnothing$;
(ii) for all $p \in G$ and $D \in M$, if D is dense below p in $\mathbb{P}$, then $G \cap D \neq \varnothing$;
(iii) for all open dense sets $D \in M$, we have $G \cap D \neq \varnothing$;
(iv) for all $D \in M$ that are maximal antichains in $\mathbb{P}$, we have $G \cap D \neq \varnothing$.

3.5 Names

Definition. Let $\mathbb{P}$ be a forcing poset. We define the class of $\mathbb{P}$-*names* $M^{\mathbb{P}}$ recursively as follows.

(i) $M^{\mathbb{P}}_0 = \varnothing$;
(ii) $M^{\mathbb{P}}_{\alpha+1} = \mathcal{P}^M(\mathbb{P} \times M^{\mathbb{P}}_\alpha)$;
(iii) at limit stages λ, $M^{\mathbb{P}}_\lambda = \bigcup_{\alpha<\lambda} M^{\mathbb{P}}_\alpha$;
(iv) $M^{\mathbb{P}} = \bigcup_{\alpha\in\mathrm{Ord}} M^{\mathbb{P}}_\alpha$.

Being a $\mathbb{P}$-name is absolute for transitive models. $\mathbb{P}$-names are denoted with overdots, such as in $\dot{x}$.

Definition. The *range* of a $\mathbb{P}$-name $\dot{x}$ is

$$\operatorname{ran}(\dot{x}) = \{\dot{y} \mid \exists p \in \mathbb{P}.\, \langle p, \dot{y} \rangle \in \dot{x}\}$$

Remark. Alternatively, by transfinite recursion on rank, we could define the class of $\mathbb{P}$-names over V in the following way. If rank $x = \alpha$, then x is a $\mathbb{P}$-name if and only if it is a relation such that for all $\langle p, \dot{y} \rangle \in x$, we have $p \in \mathbb{P}$ and $\dot{y}$ is a $\mathbb{P}$-name in V_α. Finally, $M^{\mathbb{P}} = \mathrm{V}^{\mathbb{P}} \cap M$.

Definition. The $\mathbb{P}$-*rank* of a name $\dot{x}$, written $\operatorname{rank}_{\mathbb{P}} \dot{x}$, is the least α such that $\dot{x} \subseteq \mathbb{P} \times M^{\mathbb{P}}_\alpha$.

Definition. Let $\dot{x}$ be a $\mathbb{P}$-name and G be an arbitrary subset of $\mathbb{P}$. We define the *interpretation of $\dot{x}$ by G* recursively by

$$\dot{x}^G = \left\{\dot{y}^G \mid \exists p \in G.\, \langle p, \dot{y} \rangle \in \dot{x}\right\}$$

Definition. The *forcing extension of M by G*, written $M[G]$, is

$$M[G] = \left\{\dot{x}^G \mid \dot{x} \in M^{\mathbb{P}}\right\}$$

Example. If $\varnothing \in M$, then $\varnothing^G = \varnothing$. Let

$$\dot{x} = \{\langle p, \varnothing \rangle, \langle r, \{\langle q, \varnothing \rangle\}\rangle\}$$

If $p, q, r \in G$, then

$$\begin{aligned} \dot{x}^G &= \left\{(\langle p, \varnothing \rangle)^G, (\langle r, \{\langle q, \varnothing \rangle\}\rangle)^G\right\} \\ &= \left\{\varnothing, \left\{(\langle q, \varnothing \rangle)^G\right\}\right\} \\ &= \{\varnothing, \{\varnothing\}\} \end{aligned}$$

If $p, r \notin G$, then

$$\dot{x}^G = \varnothing$$

If $r \in G$ but $p, q \notin G$, then

$$\dot{x}^G = \left\{(\langle q, \varnothing \rangle)^G\right\} = \{\varnothing\}$$

Finally, if $p \in G$ but $r \notin G$, then

$$\dot{x}^G = \{\varnothing\}$$

We aim to show the following major theorem.

Theorem (generic model theorem). Let M be a countable transitive model of ZF, let $\mathbb{P}$ be a forcing poset, and let G be a $\mathbb{P}$-generic filter. Then

(i) $M[G]$ is a transitive set;
(ii) $|M[G]| = \aleph_0$;
(iii) $M[G] \vDash \mathsf{ZF}$, and if $M \vDash \mathsf{AC}$ then $M[G] \vDash \mathsf{AC}$;
(iv) $\mathrm{Ord}^M = \mathrm{Ord}^{M[G]}$;
(v) $M \subseteq M[G]$;

(vi) $M[G]$ is the smallest countable transitive model of ZF such that $M \subseteq M[G]$ and G is a set in $M[G]$.

Countability is only needed to show the existence of a generic filter, so parts (i) and (iii)–(vi) of this theorem hold without this assumption.

3.6 Canonical names

We can prove some parts of the generic model theorem by introducing the notion of *canonical names*.

Definition. Given a forcing poset $(\mathbb{P}, \leq, \mathbb{1})$ and a set $x \in M$, we define the *canonical* name of x by

$$\check{x} = \{\langle \mathbb{1}, \check{y} \rangle \mid y \in x\}$$

The symbol $\check{x}$ is pronounced *x-check*.

Lemma. If M is a transitive model of ZF, $\mathbb{P} \in M$, and $\mathbb{1} \in G \subseteq \mathbb{P}$, then

- for all $x \in M$, $\check{x} \in M^{\mathbb{P}}$ and $\check{x}^G = x$;
- $M \subseteq M[G]$;
- $M[G]$ is transitive.

Proof. Part (i). We show $\check{x} \in M^{\mathbb{P}}$ by induction, using the definition of $\mathbb{P}$-names by transfinite recursion. Hence

$$\check{x}^G = \left\{\check{y}^G \mid y \in x\right\} = \{y \mid y \in x\} = x$$

Part (ii) follows directly from part (i).

Part (iii). Suppose that $x \in y$ and $y \in M[G]$. By definition, $y = \dot{y}^G$ for some $\mathbb{P}$-name $\dot{y}$. By construction, any element of y is of the form $\dot{z}^G$, so in particular, $x = \dot{x}^G$ for some $\mathbb{P}$-name $\dot{x} \in M^{\mathbb{P}}$. □

Remark. Even if $G \notin M$, we can still define a name for G in M. From this, it follows that if $G \notin M$, then $M[G] \neq M$.

Proposition. Let

$$\dot{G} = \{\langle p, \check{p} \rangle \mid p \in \mathbb{P}\}$$

Then $\dot{G}^G = G$.

Proof.

$$\dot{G}^G = \left\{\check{p}^G \mid p \in G\right\} = \{p \mid p \in G\} = G$$

□

3.7 Verifying the axioms: part one

We can define unordered and ordered pairs of names, with sensible interpretations.

Definition. Given $\mathbb{P}$-names $\dot{x}, \dot{y}$, let

$$\mathrm{up}(\dot{x}, \dot{y}) = \{\langle \mathbb{1}, \dot{x} \rangle, \langle \mathbb{1}, \dot{y} \rangle\}$$

and

$$\mathrm{op}(\dot{x}, \dot{y}) = \mathrm{up}(\mathrm{up}(\dot{x}, \dot{x}), \mathrm{up}(\dot{x}, \dot{y}))$$

Proposition. For $\dot{x}, \dot{y} \in M^{\mathbb{P}}$ and $\mathbb{1} \in G \subseteq \mathbb{P}$,

$$(\mathrm{up}(\dot{x}, \dot{y}))^G = \left\{\dot{x}^G, \dot{y}^G\right\}$$

and

$$(\mathrm{op}(\dot{x}, \dot{y}))^G = \langle \dot{x}^G, \dot{y}^G \rangle$$

Lemma. Suppose M is a transitive model of ZF and $\mathbb{P} \in M$ is a forcing poset. If $\mathbb{1} \in G \subseteq \mathbb{P}$, then $M[G]$ is a transitive model of extensionality, empty set, foundation, and pairing.

Lemma. Suppose that M is a transitive model of ZF and $\mathbb{P} \in M$ is a forcing poset. Let $G \subseteq \mathbb{P}$ be such that $\mathbb{1} \in G$. Then
(i) $\mathrm{rank}(\dot{x}^G) \leq \mathrm{rank}\, \dot{x}$ for all $\dot{x} \in M^{\mathbb{P}}$;
(ii) $\mathrm{Ord}^M = \mathrm{Ord}^{M[G]}$;
(iii) $|M[G]| = |M|$.

Proof. Part (i). We show this result by induction on x. $\varnothing^G = \varnothing$, and both have rank 0. We have

$$\begin{aligned}
\mathrm{rank}(\dot{x}^G) &= \sup\left\{\mathrm{rank}\, u + 1 \mid u \in \dot{x}^G\right\} \\
&\leq \sup\left\{\mathrm{rank}(\dot{y}^G) + 1 \mid \dot{y} \in \mathrm{ran}\, \dot{x}\right\} \\
&\leq \sup\{\mathrm{rank}\, \dot{y} + 1 \mid \dot{y} \in \mathrm{ran}\, \dot{x}\} \\
&\leq \sup\{\mathrm{rank}\, u + 1 \mid u \in \dot{x}\} \\
&\leq \mathrm{rank}\, \dot{x}
\end{aligned}$$

Part (ii). Since $M \subseteq M[G]$ and being an ordinal is absolute, $\mathrm{Ord}^M \subseteq \mathrm{Ord}^{M[G]}$. For the reverse inclusion, suppose $\alpha \in M[G]$ is an ordinal, and fix a name $\dot{x} \in M^{\mathbb{P}}$ such that $\alpha = \dot{x}^G$. Then α is an ordinal in the universe, so

$$\alpha = \mathrm{rank}\, \alpha \leq \mathrm{rank}\, \dot{x}$$

so since M is transitive, $\alpha \in \mathrm{Ord}^M$.

Part (iii). Since any element of $M[G]$ is of the form $\dot{x}^G$ for some $\dot{x} \in M^{\mathbb{P}} \subseteq M \subseteq M[G]$, we must have

$$|M[G]| \leq \left|M^{\mathbb{P}}\right| \leq |M| \leq |M[G]|$$

so the inequalities must be equalities. $\square$

Corollary. $M[G]$ satisfies the axiom of infinity.

Proof. $\omega \in \mathrm{Ord}^M$ so $\omega \in \mathrm{Ord}^{M[G]} \subseteq M[G]$. $\square$

Lemma. Suppose M is a transitive model of ZF, $\mathbb{P} \in M$ is a forcing poset, and $G \subseteq \mathbb{P}$ is such that $\mathbb{1} \in G$. Then if N is another transitive model of ZF with $M \subseteq N$ a definable class in N and $G \in N$, then $M[G] \subseteq N$.

Proof. We carry out the construction of $M[G]$ in N. Namely, we will show that for all $\mathbb{P}$-names $\dot{x}$, we have $\dot{x}^G \in N$, from which it follows that $M[G] \subseteq N$. We proceed by induction on x. As the axiom of empty set holds in N and it is a transitive set, $\varnothing^G = \varnothing \in N$. Moreover, since

$$M^{\mathbb{P}} = \mathrm{V}^{\mathbb{P}} \cap M \subseteq \mathrm{V}^{\mathbb{P}} \cap N = N^{\mathbb{P}}$$

if $\dot{x}$ is a $\mathbb{P}$-name of M, it must be a $\mathbb{P}$-name of M. In particular, $x \in N$. Now, suppose that for every $\langle p, \dot{y}\rangle \in \dot{x}$, we have $\dot{y}^G \in N$. Then

$$\begin{aligned}
(\dot{x}^G)^N &= \left\{\dot{y}^G \mid \exists p \in G.\, \langle p, \dot{y}\rangle \in \dot{x}\right\}^N \\
&= \left\{(\dot{y}^G)^N \mid (\exists p \in G.\, \langle p, \dot{y}\rangle \in \dot{x})^N\right\} \\
&= \left\{\dot{y}^G \mid \exists p \in G.\, \langle p, \dot{y}\rangle \in \dot{x}\right\} \\
&= \dot{x}^G
\end{aligned}$$

Thus $\dot{x}^G \in N$ as required. $\square$

To prove the generic model theorem, it now suffices to prove the remaining axioms of ZF, which are union, power set, replacement, and separation. We can prove the axiom of union now.

Lemma. Suppose M is a transitive model of ZF, $\mathbb{P} \in M$ is a forcing poset, and $G \subseteq \mathbb{P}$ is such that $\mathbb{1} \in G$. Additionally, suppose that G is a filter. Then $M[G]$ satisfies the axiom of union.

Proof. It suffices to prove that for all $a \in M[G]$, there is some $b \in M[G]$ such that $\bigcup a = b$. Fix $\dot{a} \in M^{\mathbb{P}}$ such that $\dot{a}^G = a$, and let $\dot{b}$ be the following name.

$$\dot{b} = \{\langle p, \dot{z}\rangle \mid \exists \langle q, \dot{y}\rangle \in \dot{a}.\, \exists r \in \mathbb{P}.\, \langle r, \dot{z}\rangle \in \dot{y} \wedge p \leq r, q\}$$

Observe that $\dot{b}$ is a $\mathbb{P}$-name in M: since $\dot{a}$ is a $\mathbb{P}$-name, any $\dot{y} \in \operatorname{ran} \dot{a}$ is a $\mathbb{P}$-name, so $\dot{b}$ consists of pairs $\langle p, \dot{z}\rangle$ where $p \in \mathbb{P}$ and $\dot{z} \in \operatorname{ran} \dot{y}$ for some $\dot{y} \in \operatorname{ran} \dot{a}$. Thus $\dot{z}$ is a $\mathbb{P}$-name in V. Moreover $\dot{b} \in M$ since $\dot{b} \in \mathbb{P} \times \operatorname{tcl}(\dot{a})$.

We claim that $\bigcup a \subseteq \dot{b}^G$. Let $w \in \bigcup a$, so $w \in v$ for some $v \in a$. Since $M[G]$ is transitive, we can fix names $\dot{y}, \dot{z}$ and conditions $q, r \in G$ such that

$$\dot{y}^G = v; \quad \dot{z}^G = w; \quad \langle q, \dot{y}\rangle \in \dot{a}; \quad \langle r, \dot{z}\rangle \in \dot{y}$$

As G is a filter, by directedness there is a condition $p \leq q, r$ in G. Then, by definition, $\langle p, \dot{z}\rangle \in \dot{b}$, and $w = \dot{z}^G \in \dot{b}^G$.

For the converse, we claim that $\dot{b}^G \subseteq \bigcup a$. Let $\langle p, \dot{z}\rangle \in \dot{b}^G$, so $p \in G$ and $\dot{z}^G = c$. By definition, we can fix $\langle q, \dot{y}\rangle \in \dot{a}$ and $r \in \mathbb{P}$ such that $\langle r, \dot{z}\rangle \in \dot{y}$ and $p \leq q, r$. Using the fact that G is a filter, we must have $q, r \in G$. Hence $\dot{z}^G \in \dot{y}^G$ and $\dot{y}^G \in \dot{a}^G$, so $c \in \dot{y}^G$ for some $\dot{y}^G \in a$. □

Example (motivation for genericity)**.** Note that $\mathbb{P}, G \in M[G]$. If $M[G]$ models any reasonable theory, we should have $\mathbb{P} \setminus G \in M[G]$. We will try to build a name for $\mathbb{P} \setminus G$. A natural name to consider is

$$\dot{c} = \{\langle q, \check{p}\rangle \mid p, q \in \mathbb{P}, p \perp q\}$$

Then

$$\dot{c}^G = \{p \mid \exists q \in G.\ p \perp q\}$$

If G is a filter, its elements are pairwise compatible, so $G \cap \dot{c}^G = \varnothing$. But we still need to show that $G \cup \dot{c}^G = \mathbb{P}$. For each condition p, set

$$D_p = \{q \in \mathbb{P} \mid p \perp q \vee q \leq p\}$$

It is easy to check that $D_p \in M$ is dense. Now, if G is $\mathbb{P}$-generic, we could fix some $q \in G \cap D_p$ for any given p. Then if $p \perp q$, by definition $p \in \dot{c}^G$, and if $q \leq p$, then $p \in G$ by upwards closure. From this, it follows that $G \cup \dot{c}^G = \mathbb{P}$.

In fact, we have the following.

Proposition. Let M be a countable transitive model of ZF. Then there exists a forcing poset $\mathbb{P} \in M$ and a (non-generic) filter $G \subseteq \mathbb{P}$ such that $\mathbb{P} \setminus G \notin M[G]$.

3.8 The forcing relation

To show separation, we need to show that if $\varphi(x, y)$ is a formula and $\dot{a}, \dot{b}$ are $\mathbb{P}$-names, then

$$C = \left\{\dot{z}^G \in \dot{a}^G \mid (\varphi(\dot{z}^G, \dot{b}^G))^{M[G]}\right\} \in M[G]$$

This is unclear, even for simple formulas such as $\varphi(x, y) \equiv x \notin y$. We will build a way to formally reason about $M[G]$ from within M, without having to rely on G. To do this, we will define a relation $p \Vdash \varphi$ between conditions $p \in \mathbb{P}$ and names in $\mathrm{V}^{\mathbb{P}}$. Its relativisation $(p \Vdash \varphi)^M$ will provide a way to work in M. Our aim is to define $\Vdash$ such that $p \Vdash \varphi(\dot{u})$ if and only if for every generic subset $G \subseteq \mathbb{P}$ with $p \in G$, we have $M[G] \vDash \varphi(\dot{u}^G)$.

Naively, we might say that if $\langle p, \dot{x}\rangle \in \dot{y}$ then $p \Vdash \dot{x} \in \dot{y}$. The converse cannot be made to hold. Consider $\dot{x} = \{\langle p, \varnothing\rangle\}$ where $p \neq \mathbb{1}$. Then $p \Vdash \varnothing \in \dot{x}$. Suppose $q \perp p$, then we have $q \Vdash \dot{x} = \varnothing$. Therefore, we should have $q \Vdash \dot{x} \in \check{1}$. If we enforce the converse above, we would have $\langle q, \dot{x}\rangle \in \check{1}$, which is incorrect since $\check{1} = \{\langle \mathbb{1}, \varnothing\rangle\}$. Instead, we will define the forcing relation in terms of dense sets, leveraging the fact that generics meet all dense sets.

Definition. Let $\mathbb{P}$ be a forcing poset. The $\mathbb{P}$-*forcing language* $\mathcal{FL}_{\mathbb{P}}$ is the class of logical formulas formed using the binary relation $\in$ and constant symbols from $\mathrm{V}^{\mathbb{P}}$.

Definition. Let $\mathbb{P}$ be a forcing poset and let $p \in \mathbb{P}$. Let $\dot{x}, \dot{y}, \dot{u}$ be $\mathbb{P}$-names in V. We define the *forcing relation* $p \Vdash \varphi(\dot{u})$ recursively as follows.

(i) $p \Vdash \varphi(\dot{u}) \wedge \psi(\dot{u})$ if and only if $p \Vdash \varphi(\dot{u})$ and $p \Vdash \psi(\dot{u})$;
(ii) $p \Vdash \neg\varphi(\dot{u})$ if and only if there is no $q \leq p$ such that $q \Vdash \varphi(\dot{u})$;
(iii) $p \Vdash \exists x.\, \varphi(x, \dot{u})$ if and only if the set

$$\left\{q \leq p \mid \exists \dot{x} \in \mathrm{V}^{\mathbb{P}}.\, q \Vdash \varphi(\dot{x}, \dot{u})\right\}$$

is dense below p;
(iv) $p \Vdash \dot{x} \in \dot{y}$ if and only if the set

$$\{q \leq p \mid \exists \langle r, \dot{z}\rangle \in \dot{y}.\, q \leq r \wedge (q \Vdash \dot{x} = \dot{z})\}$$

is dense below p;
(v) $p \Vdash \dot{x} \subseteq \dot{y}$ if and only if for all $\langle q_1, \dot{z}_1\rangle \in \dot{x}$, the set

$$\{r \leq p \mid r \leq q_1 \to \exists \langle q_2, \dot{z}_2\rangle \in \dot{y}.\, r \leq q_2 \wedge (r \Vdash \dot{z}_1 = \dot{z}_2)\}$$

is dense below p; and
(vi) $p \Vdash \dot{x} = \dot{y}$ if and only if $p \Vdash \dot{x} \subseteq \dot{y}$ and $p \Vdash \dot{y} \subseteq \dot{x}$.

Remark. (i) The definitions for $\subseteq$ and $=$ are defined recursively, and thus require transfinite recursion to define formally.

(ii) All of the clauses except for the existential use only absolute notions. In particular, it does not depend on M. When relativising to a model, $(p \Vdash \exists x.\, \varphi(x))^M$ precisely when the set

$$\left\{q \leq p \mid \exists \dot{x} \in M^{\mathbb{P}}.\, q \Vdash \varphi(\dot{x}, \dot{u})\right\}$$

is dense below p.

Proposition. Let p be a condition, φ be an $\mathcal{FL}_{\mathbb{P}}$-formula, and $\dot{x}_1, \dots, \dot{x}_n$ be $\mathbb{P}$-names in V. Then the following are equivalent.
(i) $p \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n)$;
(ii) for all $q \leq p$, $q \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n)$;
(iii) there is no $q \leq p$ such that $q \Vdash \neg\varphi(\dot{x}_1, \dots, \dot{x}_n)$;
(iv) the set $\{r \mid r \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n)\}$ is dense below p.

Proof. (ii) implies (iii). If (iii) did not hold, there would be some $q \leq p$ such that $q \Vdash \neg\varphi$. Then there is no $r \leq q$ such that $r \Vdash \varphi$. So in particular, $q \nVdash \varphi$, contradicting (ii).

(iii) implies (iv). Suppose that there is no $q \leq p$ such that $q \Vdash \neg\varphi$. Take $q \leq p$. Then by assumption, $q \nVdash \neg\varphi$, so there is $r \leq q$ such that $r \Vdash \varphi$, so the set is dense as required.

(i) implies (ii). We show this by induction on formula complexity.

- For atomic formulas, let $\square$ be either $\in$ or $\subseteq$. Then $p \Vdash \dot{x} \mathrel{\square} \dot{y}$ if and only if some set A is dense below p. Take $q \leq p$, then A is dense below q. Then $q \Vdash \dot{x} \mathrel{\square} \dot{y}$ as required.
- If $p \Vdash \neg\varphi$, then there is no $r \leq p$ such that $r \Vdash \varphi$. Then there is no $r \leq q$ such that $r \Vdash \varphi$, so by definition, $q \Vdash \neg\varphi$.
- If $p \Vdash \varphi \wedge \psi$ then $p \Vdash \varphi$ and $p \Vdash \psi$, so by the inductive hypothesis, $q \Vdash \varphi$ and $q \Vdash \psi$, giving $q \Vdash \varphi \wedge \psi$.

- If $p \Vdash \exists x.\, \varphi(x)$, then A is dense below p for some set A, but then A is dense below q, so $q \Vdash \exists x.\, \varphi(x)$.

(iv) implies (i). Again, we show this by induction.

- For atomic formulas, let $\square$ be either $\in$ or $\subseteq$. To prove that $p \Vdash \dot{x} \mathrel{\square} \dot{y}$, we must show that some set A is dense below p. By assumption, the set $\{r \mid r \Vdash \dot{x} \mathrel{\square} \dot{y}\}$ is dense below p. Fix $q \leq p$, then there is $r \leq q$ such that $r \Vdash \dot{x} \mathrel{\square} \dot{y}$. Hence there is some $s \leq r \leq q \leq p$ such that $s \in A$. Therefore $p \Vdash \dot{x} \mathrel{\square} \dot{y}$ as required. The proof for existentials is the same.
- Suppose that $\{r \mid r \Vdash \varphi \wedge \psi\}$ is dense below p. So $\{r \mid r \Vdash \varphi\}$ and $\{r \mid r \Vdash \psi\}$ are also dense below p. By the inductive hypothesis, $p \Vdash \varphi$ and $p \Vdash \psi$. Hence $p \Vdash \varphi \wedge \psi$.
- Suppose that $\{r \mid r \Vdash \neg\varphi\}$ is dense below p. To show $p \Vdash \neg\varphi$, we fix $q \leq p$ and suppose $q \Vdash \varphi$. By the fact that (i) implies (iii), there is no $r \leq q$ such that $r \Vdash \neg\varphi$, contradicting density of the set $\{r \mid r \Vdash \neg\varphi\}$.

$\square$

Proposition. Let $\mathbb{P}$ be a forcing poset, let $p, q \in \mathbb{P}$, and let $\dot{a}, \dot{b} \in \mathrm{V}^{\mathbb{P}}$. Then

(i) $p \Vdash \dot{a} = \dot{a}$;

(ii) if $\langle q, \dot{b} \rangle \in \dot{a}$ and $p \leq q$, then $p \Vdash \dot{b} \in \dot{a}$;

(iii) if M is a transitive model of ZF and $\mathbb{P} \in M$, then for any φ, ψ,

$$\left\{\langle q, \dot{x} \rangle \mid \langle q, \dot{x} \rangle \in \dot{a} \wedge (q \Vdash \varphi(\dot{x}))^M\right\} \in M$$

and

$$\left\{q \in \mathbb{P} \mid (q \Vdash \psi(\dot{a}))^M\right\} \in M$$

(iv) $p \Vdash \varphi \vee \psi$ if and only if

$$\{q \leq p \mid q \Vdash \varphi \text{ or } q \Vdash \psi\}$$

is dense below p;

(v) $p \Vdash \varphi \to \psi$ if and only if there is no $q \leq p$ such that $q \Vdash \varphi$ and $q \Vdash \neg\psi$;

(vi) $p \Vdash \forall x.\, \varphi(x)$ if and only if for all $\dot{x} \in \mathrm{V}^{\mathbb{P}}$, $p \Vdash \varphi(\dot{x})$;

(vii) for any φ, the set

$$\{p \in \mathbb{P} \mid p \Vdash \varphi \text{ or } p \Vdash \neg\varphi\}$$

is a dense open set;

(viii) there is no p and formula φ such that

$$p \Vdash \varphi \wedge \neg\varphi$$

3.9 The forcing theorem

Theorem (the forcing theorem)**.** Suppose M be a transitive model of ZF, $\mathbb{P} \in M$ is a forcing poset, $\varphi(u)$ is a formula, and G is $\mathbb{P}$-generic over M. Then for any $\dot{x} \in M^{\mathbb{P}}$,

(i) if $p \in G$ and $(p \Vdash \varphi(x))^M$, then $M[G] \vDash \varphi(\dot{x}^G)$; and

(ii) if $M[G] \vDash \varphi(\dot{x}^G)$, then there is a condition $p \in G$ such that $(p \Vdash \varphi(x))^M$.

Once we have shown this theorem, we will have the following result.

Corollary. Suppose that M is a countable transitive model of ZF, $\mathbb{P} \in M$ is a forcing poset, and $\varphi(u)$ is a formula. Then for any name $\dot{x} \in M^{\mathbb{P}}$,

$$(p \Vdash \varphi(\dot{x}))^M \leftrightarrow \text{for any } \mathbb{P}\text{-generic filter } G \text{ with } p \in G, M[G] \vDash \varphi(\dot{x}^G)$$

The only reason we need countability is so that every condition is contained in a generic filter.

Proof. The forward direction is part (i) of the forcing theorem. For the backward direction, suppose that $(p \nVdash \varphi(\dot{x}))^M$. Then, by definition, there is some $q \leq p$ such that $(q \Vdash \neg\varphi(\dot{x}))^M$. Let G be a $\mathbb{P}$-generic filter over M such that $q \in G$. Then, since G is upwards closed, $p \in G$. Hence $M[G] \vDash \varphi(\dot{x}^G)$ by assumption. But as $q \in G$, by the forcing theorem we obtain $M[G] \vDash \neg\varphi(\dot{x}^G)$. This contradicts part (viii) of the proposition above by the forcing theorem. $\square$

Definition. Suppose M is a countable transitive model of ZF, $\mathbb{P} \in M$ is a forcing poset, $\dot{x}_1, \dots, \dot{x}_n \in M^{\mathbb{P}}$, $p \in \mathbb{P}$, and $\varphi(v_1, \dots, v_n)$ is a formula. Then we can define a relation $\Vdash^\star_{\mathbb{P},M}$ by

$$p \Vdash^\star_{\mathbb{P},M} \varphi(\dot{x}_1, \dots, \dot{x}_n)$$

if and only if $M[G] \vDash \varphi(\dot{x}_1^G, \dots, \dot{x}_n^G)$ for all $G \subseteq \mathbb{P}$ such that $p \in G$ and G is a $\mathbb{P}$-generic filter.

Corollary. $p \Vdash \varphi \leftrightarrow p \Vdash^\star_{\mathbb{P},M} \varphi$.

We will now prove the forcing theorem.

Proof. We show the result by induction on the complexity of formulas. Note that we need to work with relativised formulas with parameters $(p \Vdash \varphi(\mathbf{v}))^M$, but this only changes the existential case, so for all other cases we will suppress the relativisation and the parameters. We write $\Psi(\varphi)$ for the claim that for any name $\dot{x} \in M^{\mathbb{P}}$, if $p \in G$ and $(p \Vdash \varphi(\dot{x}))^M$, then $M[G] \vDash \varphi(\dot{x}^G)$, and if $M[G] \vDash \varphi(\dot{x}^G)$, then there exists $p \in G$ such that $(p \Vdash \varphi(\dot{x}))^M$.

Part (i): negations. Suppose $\Psi(\varphi)$ holds. Let $p \in G$ and $p \Vdash \neg\varphi$. Suppose for a contradiction that $M[G] \vDash \varphi$, or equivalently, $\varphi^{M[G]}$. Then as $\Psi(\varphi)$ holds, there is $q \in G$ such that $q \Vdash \varphi$. As G is a filter, there is $r \in G$ such that $r \leq p, q$. Then $r \Vdash \varphi$, which contradicts the definition of $p \Vdash \neg\varphi$. Hence $\neg(\varphi^{M[G]})$, so by definition $(\neg\varphi)^{M[G]}$, so $M[G] \vDash \neg\varphi$.

For the converse, suppose $M[G] \vDash \neg\varphi$. Let

$$D = \{p \in \mathbb{P} \mid p \Vdash \varphi \vee p \Vdash \neg\varphi\}$$

Then D is dense, because if $q \nVdash \varphi$, then there is $p \leq q$ such that $p \Vdash \neg\varphi$, and $p \in D$. So as G is generic, we can fix $p \in G \cap D$. If $p \Vdash \varphi$, then by $\Psi(\varphi)$ we must have $M[G] \vDash \varphi$, but we assumed $M[G] \vDash \neg\varphi$. Hence $p \Vdash \neg\varphi$.

Part (ii): conjunctions. Suppose $\Psi(\varphi)$ and $\Psi(\psi)$. Suppose $p \Vdash \varphi \wedge \psi$ for some $p \in G$, so by definition, $p \Vdash \varphi$ and $p \Vdash \psi$. By $\Psi(\varphi)$ and $\Psi(\psi)$, we have $M[G] \vDash \varphi$ and $M[G] \vDash \psi$. So $M[G] \vDash \varphi \wedge \psi$.

For the converse, suppose $M[G] \vDash \varphi \wedge \psi$. Then $M[G] \vDash \varphi$ and $M[G] \vDash \psi$, so there are $p, q \in G$ such that $p \Vdash \varphi$ and $q \Vdash \psi$. But G is a filter, so there is $r \leq p, q$ such that $r \Vdash \varphi$ and $r \Vdash \psi$. Hence $r \Vdash \varphi \wedge \psi$, as required.

Part (iii): existential quantifiers. For this case, we will not suppress relativisation and parameters. Suppose $\Psi(\varphi(\dot{x}))$; we show $\Psi(\exists x.\, \varphi(x))$. To be more precise, for all names $\dot{x} \in M^{\mathbb{P}}$, we assume the forcing theorem holds for $\varphi(\dot{x})$. Suppose $p \in G$ is such that $(p \Vdash \exists x.\, \varphi(x))^M$. Let

$$D = \left(\left\{q \leq p \mid \exists \dot{x} \in \mathrm{V}^{\mathbb{P}}.\, (q \Vdash \varphi(\dot{x}))\right\}\right)^M = \left\{q \leq p \mid \exists \dot{x} \in M^{\mathbb{P}}.\, (q \Vdash \varphi(\dot{x}))^M\right\} \in M$$

By definition of forcing existentials, D is a dense set. Since G is generic, there is some $q \in G \cap D$. Then we can fix some $\mathbb{P}$-name $\dot{x}$ such that $(q \Vdash \varphi(\dot{x}))^M$. Since the forcing theorem holds for $\varphi(\dot{x})$, we have $M[G] \vDash \varphi(\dot{x}^G)$. Hence $M[G] \vDash \exists x.\, \varphi(x)$.

Now suppose $M[G] \vDash \exists x.\, \varphi(x)$. We can fix $\dot{x} \in M^{\mathbb{P}}$ such that $M[G] \vDash \varphi(\dot{x}^G)$. By the fact that $\Psi(\varphi(\dot{x}))$ holds, there is a condition p such that $(p \Vdash \varphi(\dot{x}))^M$. Then

$$\left\{q \leq p \mid (q \Vdash \varphi(\dot{x}))^M\right\}$$

is dense. Hence, by definition, $(p \vDash \exists x.\, \varphi(x))^M$.

Part (iv): equality. Recall that $p \Vdash \dot{x} = \dot{y}$ if and only if

(a) for all $\langle q_1, \dot{z}_1\rangle \in \dot{x}$, $\{r \leq p \mid r \leq q_1 \to \exists \langle q_2, \dot{z}_2\rangle \in \dot{y}.\, r \leq q_2 \wedge (r \Vdash \dot{z}_1 = \dot{z}_2)\}$ is dense below p; and

(b) for all $\langle q_2, \dot{z}_2\rangle \in \dot{y}$, $\{r \leq p \mid r \leq q_2 \to \exists \langle q_1, \dot{z}_1\rangle \in \dot{x}.\, r \leq q_1 \wedge (r \Vdash \dot{z}_1 = \dot{z}_2)\}$ is dense below p.

We show that for any $\dot{x}, \dot{y}$, we have $\Psi(\dot{x} = \dot{y})$. We will show this by transfinite induction on the pair $\langle \dot{x}, \dot{y}\rangle$ ordered lexicographically.

Suppose that $p \Vdash \dot{x} = \dot{y}$ and $p \in G$. We show $M[G] \vDash \dot{x}^G \subseteq \dot{y}^G$; the converse holds by symmetry, and then we obtain $M[G] \vDash \dot{x}^G = \dot{y}^G$ by extensionality. Any element of $\dot{x}^G$ is of the form $\dot{z}_1^G$ where $\langle q_1, \dot{z}_1\rangle \in \dot{x}$ and $q_1 \in G$. Since G is a filter, we can fix $s \in G$ such that $s \leq p, q_1$. Then, as $s \leq p$, we have $s \Vdash \dot{x} = \dot{y}$, so the set in (a) above is dense below s. Hence there is $r \in G$ such that $r \leq s \leq q_1$ and there exists $\langle q_2, \dot{z}_2\rangle \in \dot{y}$ such that $r \leq q_2$ and $r \Vdash \dot{z}_1 = \dot{z}_2$. As G is a filter, $q_2 \in G$, so $\dot{z}_2^G \in \dot{y}^G$. By using the inductive hypothesis on $\langle \dot{z}_1, \dot{z}_2\rangle$, as $r \in G$ we have $M[G] \vDash \dot{z}_1^G = \dot{x}_2^G$. Hence $\dot{z}_1^G \in \dot{y}^G$, so $\dot{x}^G \subseteq \dot{y}^G$.

For the converse, $M[G] \vDash \dot{x}^G = \dot{y}^G$. Define D to be the set of $r \in \mathbb{P}$ such that at least one of the following hold.

(0) $r \Vdash \dot{x} = \dot{y}$;

(a′) there exists $\langle q_1, \dot{z}_1\rangle \in \dot{x}$ such that $r \leq q_1$ and for all $\langle q_2, \dot{z}_2\rangle \in \dot{y}$ and $s \in \mathbb{P}$, if $s \leq q_2$ and $s \Vdash \dot{z}_1 = \dot{z}_2$ then $s \perp r$;

(b′) there exists $\langle q_2, \dot{z}_2\rangle \in \dot{y}$ such that $r \leq q_2$ and for all $\langle q_1, \dot{z}_1\rangle \in \dot{x}$ and $s \in \mathbb{P}$, if $s \leq q_1$ and $s \Vdash \dot{z}_1 = \dot{z}_2$ then $s \perp r$.

Note that by separation in M and absoluteness, D is a set in M. We claim that D is dense. Fix $p \in \mathbb{P}$, and suppose $p \nVdash \dot{x} = \dot{y}$. Then at least one of (a) and (b) above fails. Suppose that the set in (a) fails; the result for (b) holds by symmetry. Then there is $\langle q_1, \dot{z}_1\rangle \in \dot{x}$ such that

$$\{r \leq p \mid r \leq q_1 \to \exists \langle q_2, \dot{z}_2\rangle \in \dot{y}.\, r \leq q_2 \wedge (r \Vdash \dot{z}_1 = \dot{z}_2)\}$$

is not dense below p. Then there is $s \leq p$ such that for all $r \leq s$, we have $r \leq q_1$, and for all $\langle q_2, \dot{z}_2\rangle \in \dot{y}$ such that $\neg((r \Vdash \dot{z}_1 = \dot{z}_2) \wedge r \leq q_2)$. In particular, this gives $s \leq q_1$. Now, if $\langle q_1, \dot{z}_2\rangle \in \dot{y}$, $r \leq q_2$, and $r \Vdash \dot{z}_1 = \dot{z}_2$, then it must be the case that $s \perp r$, as any common extension of s and r would contradict the fact that the set in (a) was not dense. Thus $s \leq p$ and s satisfies (a′). Hence D is dense.

D is dense below $p \in G$ and G is $\mathbb{P}$-generic so we can fix $r \in G \cap D$. We will show that r satisfies (0), which finishes the proof. Suppose not, so suppose r satisfies (a′) without loss of generality. Then we can fix $\langle q_1, \dot{z}_1 \rangle \in \dot{x}$ such that $r \leq q_1$ and for all $\langle q_2, \dot{z}_2 \rangle \in \dot{y}$ such that for all $s \in \mathbb{P}$ with $s \leq q_2$ and $s \Vdash \dot{z}_1 = \dot{z}_2$, we have $s \perp r$. Since $r \in G$ and $r \leq q_1$, we must have $q_1 \in G$ by upwards closure. Therefore, $M[G] \vDash \dot{z}_1^G \in \dot{x}^G = \dot{y}^G$. So we can fix $\langle q_2, \dot{z}_2 \rangle \in \dot{y}$ such that $q_2 \in G$ and $M[G] \vDash \dot{z}_1^G = \dot{z}_2^G$. By the inductive hypothesis, we can fix $p' \in G$ such that $p' \Vdash \dot{z}_1 = \dot{z}_2$. Since G is a filter and both $p', q_2 \in G$, we obtain $s \in G$ with $s \leq p', q_2$. Hence $s \Vdash \dot{z}_1 = \dot{z}_2$. Hence, by (a′), we have $s \perp r$. But $s, r \in G$, so $s \parallel r$, giving a contradiction.

Part (v): membership. Suppose that $p \Vdash \dot{x} \in \dot{y}$ for $p \in G$. Let

$$D = \{q \leq p \mid \exists \langle r, \dot{z} \rangle \in \dot{y}.\, q \leq r \wedge (q \Vdash \dot{x} = \dot{z})\}$$

By definition, D is dense. We can fix $q \in G \cap D$. Since $q \in D$, we may also fix $\langle r, \dot{z} \rangle \in \dot{y}$ such that $q \leq r$ and $q \Vdash \dot{x} = \dot{z}$. As $q \in G$, by the forcing theorem for equality, $M[G] \vDash \dot{x}^G = \dot{z}^G$. Since G is a filter and $q \leq r$, then $r \in G$ and so $\dot{z}^G \in \dot{y}^G$. Hence $M[G] \vDash \dot{x}^G \in \dot{y}^G$.

Now suppose $M[G] \vDash \dot{x}^G \in \dot{y}^G$. Fix $\langle r, \dot{z} \rangle \in \dot{y}$ such that $r \in G$ and $\dot{z}^G = \dot{x}^G$. Now, by the forcing theorem for equality, there is $q \in G$ such that $q \Vdash \dot{x} = \dot{z}$. Since G is a filter and $q, r \in G$, we can fix $p \in G$ such that $p \leq q, r$. Then $p \Vdash \dot{z} \in \dot{y}$ and $p \Vdash \dot{x} = \dot{z}$. So for all $s \leq p$, we have $s \leq r$ and $s \Vdash \dot{x} = \dot{z}$, so D is dense below p. Hence $p \Vdash \dot{x} \in \dot{y}$, as required. ☐

3.10 Verifying the axioms: part two

Lemma. Suppose that M is a countable transitive model of ZF, $\mathbb{P} \in M$ is a forcing poset, and $G \subseteq \mathbb{P}$ is a generic filter. Then $M[G]$ models separation.

Proof. Let $\varphi(x, \upsilon)$ be a formula with free variables x, υ. It suffices to show that for any $a, \upsilon \in M[G]$,

$$b = \{x \in a \mid M[G] \vDash \varphi(x, \upsilon)\} \in M[G]$$

Fix names $\dot{a}, \dot{\upsilon}$ such that $\dot{a}^G = a$ and $\dot{\upsilon}^G = \upsilon$. Any member of $\dot{a}^G$ is of the form $\dot{x}^G$ where $\langle p, \dot{x} \rangle \in \dot{a}$ and $p \in G$. Then

$$b = \left\{\dot{x}^G \mid \exists p \in G.\, \langle p, \dot{x} \rangle \in \dot{a} \wedge M[G] \vDash \varphi(\dot{x}^G, \dot{\upsilon}^G)\right\}$$

We define

$$\dot{b} = \left\{\langle p, \dot{x} \rangle \mid \langle p, \dot{x} \rangle \in \dot{a} \wedge (p \Vdash \varphi(\dot{x}, \dot{\upsilon}))^M\right\} \in M^{\mathbb{P}}$$

Thus, $\dot{b}^G \in M[G]$, so it suffices to show $\dot{b}^G = b$. We have $x \in \dot{b}^G$ if and only if there is some $\mathbb{P}$-name $\dot{x}$ in M and $p \in G$ such that $\dot{x}^G = x$, $\langle p, \dot{x} \rangle \in \dot{a}$, and $(p \Vdash \varphi(\dot{x}, \dot{\upsilon}))^M$. By the forcing theorem, this is equivalent to the statement $x \in \dot{a}^G$ and $M[G] \vDash \varphi(x, \upsilon)$, which is precisely the statement $x \in b$. ☐

The arguments for collection and power set will follow the same pattern.

Lemma. Suppose that M is a countable transitive model of ZF, $\mathbb{P} \in M$ is a forcing poset, and $G \subseteq \mathbb{P}$ is a generic filter. Then $M[G]$ models collection.

Proof. Let $\varphi(x, y, v)$ be a formula with free variables x, y, v. Fix $a, v \in M[G]$ with names $\dot{a}, \dot{v}$. Suppose $M[G] \vDash \forall x \in a. \exists y. \varphi(x, y, v)$. We claim that there is $b \in M[G]$ such that $M[G] \vDash \forall x \in a. \exists y \in b. \varphi(x, y, v)$. Let

$$C = \left\{\langle p, \dot{x}\rangle \mid p \in \mathbb{P} \wedge \dot{x} \in \operatorname{ran} \dot{a} \wedge \exists \dot{y} \in M^{\mathbb{P}}. (p \Vdash \varphi(\dot{x}, \dot{y}, \dot{v}))^M\right\}$$

Then for all $\langle p, \dot{x}\rangle \in C$, there is $\dot{y} \in M^{\mathbb{P}}$ such that $(p \Vdash \varphi(\dot{x}, \dot{y}, \dot{v}))^M$. Note that the collection of such $\dot{y}$ might not form a set, for example with the formula $\varphi(x, y) \equiv x \in y$. However, using collection in M, we may form a set $B \in M$ such that $B \subseteq M^{\mathbb{P}}$ and

$$\forall \langle p, \dot{x}\rangle \in C. \exists \dot{y} \in B. (p \Vdash \varphi(\dot{x}, \dot{y}, \dot{v}))^M$$

Finally, set

$$\dot{b} = \{\langle \mathbb{1}, \dot{y}\rangle \mid \dot{y} \in B\} \in M^{\mathbb{P}}$$

We show that $b = \dot{b}^G$ satisfies the required property. Fix some $x \in a$, then by definition there is $\langle q, \dot{x}\rangle \in \dot{a}$ such that $q \in G$ and $\dot{x}^G = x$. By assumption, $M[G] \vDash \exists y \in b. \varphi(x, y, v)$. So fix $\dot{z}^G$ such that $M[G] \vDash \varphi(x, z, v)$. By the forcing theorem, there is $p \in G$ such that $(p \Vdash \varphi(\dot{x}, \dot{z}, \dot{v}))^M$. Hence $\langle p, \dot{x}\rangle \in C$. So we can fix $\dot{y} \in B$ such that $(p \Vdash \varphi(\dot{x}, \dot{y}, \dot{v}))^M$. Therefore, $\langle \mathbb{1}, \dot{y}\rangle \in \dot{b}$. Since $\mathbb{1} \in G$, $\dot{y}^G \in \dot{b}^G$. By the forcing theorem again,

$$M[G] \vDash \dot{y}^G \in \dot{b}^G \wedge \varphi(\dot{x}^G, \dot{y}^G, v)$$

Hence, collection holds. $\square$

Note that since power set has not been used in any of the previous proofs, if $M \vDash \mathsf{ZF}^-$, then $M[G] \vDash \mathsf{ZF}^-$.

Lemma. Suppose that M is a countable transitive model of ZF, $\mathbb{P} \in M$ is a forcing poset, and $G \subseteq \mathbb{P}$ is a generic filter. Then $M[G]$ models the axiom of power set.

Proof. By separation, it suffices to show that if $a \in M[G]$, then

$$\mathcal{P}(a) \cap M[G] = \{x \in M[G] \mid x \subseteq a\} \subseteq b$$

for some set $b \in M[G]$. Fix $a \in M[G]$ with name $\dot{x} \in M^{\mathbb{P}}$, and define

$$S = \left\{\dot{x} \in M^{\mathbb{P}} \mid \operatorname{ran} \dot{x} \subseteq \operatorname{ran} \dot{a}\right\} = \mathcal{P}(\mathbb{P} \times \operatorname{ran} \dot{a})^M$$

and let

$$\dot{b} = \{\langle \mathbb{1}, \dot{x}\rangle \mid x \in S\} \in M^{\mathbb{P}}$$

Let $c \in \mathcal{P}(a) \cap M[G]$; we must show that $c \in \dot{b}^G$. Let $\dot{c} \in M^{\mathbb{P}}$ be a name for c, and let

$$\dot{x} = \left\{\langle p, \dot{z}\rangle \mid \dot{z} \in \operatorname{ran} \dot{a} \wedge (p \Vdash \dot{z} \in \dot{c})^M\right\} \in S$$

We claim $\dot{x}^G = \dot{c}^G = c$. First, we show $\dot{x}^G \subseteq c$. Fix $\dot{z}^G \in \dot{x}^G$. By definition, we can fix $p \in G$ such that $\langle p, \dot{z}\rangle \in \dot{x}$. From this, it follows that $\dot{z} \in \operatorname{ran} \dot{a}$ and $p \Vdash \dot{z} \in \dot{c}$. Since $p \in G$, by the forcing theorem, $M[G] \vDash \dot{z}^G \in \dot{c}^G$, as required.

Conversely, since $M[G] \vDash c \subseteq \dot{a}^G$, every element of c is of the form $\dot{z}^G$ for $\langle q, \dot{z}\rangle \in \dot{a}$ with $q \in G$. Also, if $M[G] \vDash \dot{z}^G \in c$, then by the forcing theorem, there is p such that $p \Vdash \dot{z} \in \dot{c}$. Then $\langle p, \dot{z}\rangle \in \dot{x}$, so $\dot{z}^G \in \dot{x}^G$. $\square$

Lemma. Suppose that M is a countable transitive model of ZFC^-, $\mathbb{P} \in M$ is a forcing poset, and $G \subseteq \mathbb{P}$ is a generic filter. Then $M[G]$ models the well-ordering principle, and hence models ZFC^-.

Proof. It suffices to show that any $a \in M[G]$ can be well-ordered in $M[G]$. Fix a name $\dot{a}$ for a. Using the well-ordering principle in M, we can enumerate the elements of ran $\dot{a}$ as

$$\{\dot{x}_\alpha \mid \alpha < \delta\}$$

Let

$$\dot{f} = \{\langle \mathbb{1}, \mathrm{op}(\check{\alpha}, \dot{x}_\alpha)\rangle \mid \alpha < \delta\} \in M^{\mathbb{P}}$$

So in $M[G]$,

$$\dot{f}^G = \left\{\langle \alpha, \dot{x}_\alpha^G\rangle \mid \alpha < \delta\right\}$$

Hence $\dot{f}^G$ is a function with domain δ, and $a \subseteq \operatorname{ran} \dot{f}^G$. We can now define a well-order $\prec$ on a by defining that $x \prec y$ if and only if

$$\min\left\{\alpha < \delta \mid \dot{f}^G(\alpha) = x\right\} < \min\left\{\alpha < \delta \mid \dot{f}^G(\alpha) = y\right\}$$

□

Remark.

(i) $\dot{f}^G$ may not be injective, since we could have $\dot{x}_\alpha^G = \dot{x}_\beta^G$ for $\alpha \neq \beta$.

(ii) ran $\dot{f}^G$ may not equal a. Elements of $\dot{a}$ are conditions $\langle p, \dot{x}_\alpha\rangle$, and if $p \notin G$, we may not have $\dot{x}_\alpha^G \in a$.

(iii) For power set, it sufficed to find a set of names which contained enough names to represent all possible subsets of a. However, there are a proper class of names for the empty set, so we could not produce a set of all such names.

(iv) The statement $M[G] \vDash \varphi$ should be considered a ternary relation between M, G, and φ. It is possible that G and H are both generic, but $M[G] \vDash \varphi$ and $M[H] \vDash \neg\varphi$.

(v) The relativisation $(p \Vdash \varphi)^M$ will be dropped when clear in subsequent sections.

Lemma. Let M be a countable transitive model of ZFC and let $\mathbb{P} \in M$ be a forcing poset. Let φ, ψ be $\mathcal{FL}_{\mathbb{P}}$-formulas. Then, for any $p \in \mathbb{P}$ and $\dot{x} \in M^{\mathbb{P}}$,

(i) if $\mathsf{ZFC} \vdash \forall v.\, \varphi(v) \to \psi(v)$ then $(p \Vdash \varphi(\dot{x}))^M \to (p \Vdash \psi(\dot{x}))^M$; and

(ii) if $\mathsf{ZFC} \vdash \forall v.\, \varphi(v) \leftrightarrow \psi(v)$ then $(p \Vdash \varphi(\dot{x}))^M \leftrightarrow (p \Vdash \psi(\dot{x}))^M$.

Informally, forcing is closed under logical equivalence.

Proof. Clearly (ii) follows from (i). Suppose that $\mathsf{ZFC} \vdash \forall v.\, \varphi(v) \to \psi(v)$ and $(p \Vdash \varphi(\dot{x}))^M$. Since M is countable, we can let G be a $\mathbb{P}$-generic filter over M such that $p \in G$. By the forcing theorem, $M[G] \vDash \varphi(\dot{x}^G)$. Since $M[G] \vDash \mathsf{ZFC}$, we have $M[G] \vDash \psi(\dot{x}^G)$. Hence, by the forcing theorem in the reverse direction, as this is true for all generics containing p we have $(p \Vdash \psi(\dot{x}))^M$. □

4 Forcing and independence results

4.1 Independence of the constructible universe

In this subsection, we show Con(ZFC + V $\neq$ L), and thus V $\neq$ L is independent of the axioms of ZFC.

Theorem. Let M be a countable transitive model of ZFC. Then there is a countable transitive model $N \supseteq M$ such that $N \vDash \mathsf{ZFC} + \mathrm{V} \neq \mathrm{L}$.

Proof. Let M be a countable transitive model of ZFC, and let $\mathbb{P} \in M$ be any atomless forcing poset (that is, it has no minimal elements), for example $\mathrm{Fn}(\omega, 2)$. Since M is countable, we can let G be a $\mathbb{P}$-generic filter over M. As $\mathbb{P}$ is atomless, $G \notin M$. Hence $M \subsetneq M[G] \vDash \mathsf{ZFC}$.

We show that $M[G] \vDash \mathrm{V} \neq \mathrm{L}$. We have

$$\mathrm{L}_{\mathrm{Ord} \cap M} = \mathrm{L}^M \subseteq M \subsetneq M[G]$$

By the generic model theorem, $\mathrm{Ord} \cap M = \mathrm{Ord} \cap M[G]$, so $M[G] \neq \mathrm{L}_{\mathrm{Ord} \cap M[G]} = \mathrm{L}^{M[G]}$. In particular, we have $(\mathrm{V} \neq \mathrm{L})^{M[G]}$. □

We will now discuss how to remove the assumption that we have a countable transitive model of ZFC.

Theorem. If Con(ZFC), then Con(ZFC + V $\neq$ L). Hence, ZFC $\nvdash$ V = L.

Proof. Suppose that ZFC + V $\neq$ L gives rise to a contradiction. Then, from a finite set of axioms $\Gamma \subseteq \mathsf{ZFC} + \mathrm{V} \neq \mathrm{L}$, we can find ψ such that $\Gamma \vdash \psi \wedge \neg\psi$. By following the previous proofs, there is a finite set of axioms $\Lambda \subseteq \mathsf{ZFC}$ such that ZFC proves that if there is a countable transitive model of Λ, then there is a countable transitive model of Γ. This set Λ should be sufficient to do the following:

(i) to prove basic properties of forcing and constructibility;

(ii) to prove the necessary facts about absoluteness, such as absoluteness of finiteness, partial orders and so on;

(iii) to prove facts about forcing, including the forcing theorem; and

(iv) if M is a countable transitive model of Λ with $\mathbb{P} \in M$ and G is $\mathbb{P}$-generic over M, then Λ proves that $M[G] \vDash \Gamma$.

As Λ is finite and a subset of the axioms of ZFC, then by the reflection theorem there is a countable transitive model of Λ. Hence, there is a countable transitive model N of Γ. But $\Gamma \vdash \psi \wedge \neg\psi$, so $N \vDash \psi \wedge \neg\psi$. Hence $(\psi \wedge \neg\psi)^N$, so in ZFC we can prove $\psi^N \wedge \neg\psi^N$, so ZFC is inconsistent. □

Remark. Gunther, Pagano, Sánchez Terraf, and Steinberg recently completed a formalisation of the countable transitive model approach to forcing in the interactive theorem prover Isabelle. To obtain Con(ZFC) → Con(ZFC + ¬CH), they used ZC together with 21 instances of replacement, which are explicitly enumerated in the paper.

4.2 Cohen forcing

Fix a countable transitive model M of ZFC. Recall that for $I, J \in M$,

(i) $\mathrm{Fn}(I, J) = \{p \mid p \text{ is a finite partial function } I \to J\}$, together with $\supseteq$ and $\varnothing$, has the structure of a forcing poset.

(ii) $\mathrm{Fn}(I, J)$ is always a set in M.

(iii) $\mathrm{Fn}(I, J)$ has the countable chain condition if and only if I is empty or J is countable.

(iv) The sets $D_i = \{q \in \mathrm{Fn}(I, J) \mid i \in \operatorname{dom} q\}$ and $R_j = \{q \in \mathrm{Fn}(I, J) \mid i \in \operatorname{ran} q\}$ are dense for all $i \in I$ and $j \in J$.

Now, suppose that $G \subseteq \mathrm{Fn}(I, J)$ is generic over M. Since G is a filter, if $p, q \in G$ then $p \cap q \in G$. Hence, if $p, q \in G$, then p, q agree on the intersection of their domains. Let $f_G = \bigcup G$. Then f_G is a function with domain contained in I and range contained in J. Note that this function has name

$$\dot{f} = \{\langle p, \mathrm{op}(\check{\imath}, \check{\jmath})\rangle \mid p \in \mathbb{P}, \langle i, j\rangle \in p\}$$

Since D_i, R_j are dense, we obtain $G \cap D_i \neq \varnothing$, so we must have $i \in \operatorname{dom} f_G$. Similarly, $j \in \operatorname{ran} f_G$. We therefore obtain the following.

Proposition. Let $G \subseteq \mathrm{Fn}(I, J)$ be a generic filter over M, and suppose I, J are nonempty. Then $M[G] \vDash f_G : I \to J$ is a surjection.

Proposition. Suppose that I, J are nonempty sets, at least one of which is infinite. Then

$$|\mathrm{Fn}(I, J)| = \max(|I|, |J|)$$

In particular, $|\mathrm{Fn}(\omega, 2)| = \aleph_0$.

Proof. Each condition $p \in \mathrm{Fn}(I, J)$ is a finite function, so from this it follows that

$$\mathrm{Fn}(I, J) \subseteq (I \times J)^{<\omega}$$

Hence

$$\mathrm{Fn}(I, J) \subseteq |(I \times J)^{<\omega}| = |I \times J| = \max(|I|, |J|)$$

For the reverse direction, if we fix $i_0 \in I$ and $j_0 \in J$, then

$$\{\langle i_0, j\rangle \mid j \in J\} \cup \{\langle i, j_0\rangle \mid i \in I\}$$

is a collection of $|I \cup J|$-many distinct elements of $\mathrm{Fn}(I, J)$. Thus

$$\max(|I|, |J|) = |I \cup J| \leq \mathrm{Fn}(I, J)$$

as required. $\square$

We aim to provide a model in which CH fails. To do this, we will consider the forcing poset $\mathrm{Fn}(\omega_2^M \times \omega, 2)$. We may consider $f_G : \omega_2^M \times \omega \to 2$, and let $g_\alpha : \omega \to 2$ be the function defined by $g_\alpha(n) =$

$f_G(\alpha, n)$. This provides ω_2^M-many reals in $M[G]$. To show that $M[G] \vDash \mathsf{ZFC} + \neg\mathsf{CH}$, we must show that all of the g_α are distinct, and that

$$\omega_1^{M[G]} = \omega_1^M; \quad \omega_2^{M[G]} = \omega_2^M$$

It will turn out that the countable chain condition guarantees that all cardinals in M remain cardinals in $M[G]$.

Example. Let κ be an uncountable cardinal in M, and consider $\mathrm{Fn}(\omega, \kappa)$, which does not satisfy the countable chain condition. Then in $M[G]$, the function $f_G : \omega \to \kappa$ is a surjection. Hence, κ has been collapsed into a countable ordinal in $M[G]$.

4.3 Preservation of cardinals

Definition. Let $\mathbb{P} \in M$ be a forcing poset. We say that $\mathbb{P}$ *preserves cardinals* if and only if for every generic filter $G \subseteq \mathbb{P}$ over M and every $\kappa \in \mathrm{Ord} \cap M$,

$$(\kappa \text{ is a cardinal})^M \leftrightarrow (\kappa \text{ is a cardinal})^{M[G]}$$

Also, $\mathbb{P}$ *preserves cofinalities* if and only if for every generic filter $G \subseteq \mathbb{P}$ over M,

$$\mathrm{cf}^M(\gamma) = \mathrm{cf}^{M[G]}(\gamma)$$

for all limit ordinals γ.

Recall that being a cardinal is Π_1-definable so downwards absolute. In particular, cardinals of $M[G]$ are automatically cardinals of M. Also, note that finiteness and being ω are absolute.

Lemma. Let $\mathbb{P} \in M$ be a forcing poset. Then
(i) $\mathbb{P}$ preserves cofinalities if and only if for every generic filter G, for all limit ordinals β with $\omega < \beta < \mathrm{Ord} \cap M$,

$$(\beta \text{ is regular})^M \to (\beta \text{ is regular})^{M[G]}$$

and
(ii) if $\mathbb{P}$ preserves cofinalities, then $\mathbb{P}$ preserves cardinals.

The converse of (ii) is not true. Note that the definition of regularity did not require being a cardinal, but is a consequence.

Proof. Part (i). Suppose $\mathbb{P}$ preserves cofinalities and G is $\mathbb{P}$-generic. Fix a limit ordinal β such that $\omega < \beta < \mathrm{Ord} \cap M$. Then if β is regular in M, we have

$$\beta = \mathrm{cf}^M(\beta) = \mathrm{cf}^{M[G]}(\beta)$$

Hence β is regular in $M[G]$. Conversely, suppose γ is a limit ordinal such that $\omega < \gamma < \mathrm{Ord} \cap M$. Let $\beta = \mathrm{cf}^M(\gamma)$. Then β is a regular cardinal in M. Let $f \in M$ be a strictly increasing cofinal function $\beta \to \gamma$. If β is uncountable in M, then β is regular in $M[G]$ by assumption. Otherwise, $\beta = \omega$, and then $\beta = \omega^{M[G]}$ by absoluteness, and so again β is regular in $M[G]$. As $f \in M$, also $f \in M[G]$, so there is a strictly increasing cofinal map $\beta \to \gamma$ in $M[G]$, so

$$\mathrm{cf}^{M[G]}(\gamma) = \mathrm{cf}^{M[G]}(\beta) = \beta = \mathrm{cf}^{M}(\gamma)$$

Part (ii). Suppose that $\mathbb{P}$ preserves cofinalities. Let κ be a cardinal in M. One of three cases occur.

(a) If $\kappa \leq \omega$, then $(\kappa \leq \omega)^{M[G]}$, so κ is a cardinal in $M[G]$;

(b) If κ is regular in M, then κ is regular in $M[G]$ by (i), so it is a cardinal in $M[G]$.

(c) Suppose κ is singular in M. In this case, one can show that κ is the supremum of a set S of regular cardinals in M. One way to show this is that if κ is the supremum of a set T of cardinals, we can set $S = \{\lambda^+ \mid \lambda \in T\}$. Since $\mathbb{P}$ preserves regular cardinals, every element of S is regular in $M[G]$, and in particular they are cardinals. Hence κ is the supremum of a set of cardinals, and is therefore a cardinal.

□

Lemma (the approximation lemma)**.** Let $A, B, \mathbb{P} \in M$, and suppose that ($\mathbb{P}$ has the countable chain condition)^M. Let G be $\mathbb{P}$-generic over M. Then for any function $f \in M[G]$ with $f : A \to B$, there is a function $F \in M$ with $F : A \to \mathcal{P}^M(B)$ such that for all $a \in A$, we have $f(a) \in F(a)$ and $(|F(a)| \leq \aleph_0)^M$.

This proof requires that M is countable. Note that the relativisation of the countable chain condition to M ensures that the hypothesis is non-vacuous, as any forcing poset in M is externally countable.

Proof. Suppose that $M[G] \vDash f : A \to B$. Since $A, B \in M$, we have canonical names $\check{A}, \check{B} \in M^{\mathbb{P}}$. Let $\dot{f}$ be a name for f. By the forcing theorem, there is a condition $p \in G$ such that

$$p \Vdash \dot{f} : \check{A} \to \check{B} \text{ is a function}$$

Define $F : A \to \mathcal{P}^M(B)$ by

$$F(a) = \left\{b \in B \mid \exists q \leq p.\, q \Vdash \dot{f}(\check{a}) = \check{b}\right\}$$

Note that $F(a) \in M$ by the definability of the forcing relation, so as $A \in M$, the set

$$F = \{\langle a, F(a)\rangle \mid a \in A\}$$

is a set in M. We now show that this definition has the desired properties. Observe that as F is a function in M, it is also a function in V. We show that $f(a) \in F(a)$. Suppose that $M[G] \vDash f(a) = b$ for $b \in B$. By the forcing theorem, there is $q \in G$ such that $q \Vdash \dot{f}(\check{a}) = \check{b}$. As G is a filter, there is $r \leq p, q$ with $r \in G$ witnessing $b \in F(a)$ as required.

We now show that $|F(a)| \leq \aleph_0$. Working in M, and in particular using the axiom of choice in M, for each $b \in F(a)$ there is a condition $q_b \leq p$ such that $q_b \Vdash \dot{f}(\check{a}) = \check{b}$. It suffices to show that $q_b \perp q_c$ for $b \neq c$, because then they form an antichain, so by the countable chain condition we may conclude $|F(a)| \leq \aleph_0$. Suppose not, so let $r \leq q_b, q_c$. Then

$$r \Vdash \dot{f} : \check{A} \to \check{B} \text{ is a function} \wedge \dot{f}(\check{a}) = \check{b} \wedge \dot{f}(\check{a}) = \check{c} \wedge \check{b} \neq \check{c}$$

Let H be a generic filter with $r \in H$; this exists by countability of M. Then $r \leq p$ and

$$M[H] \vDash f : A \to B \text{ is a function} \wedge f(a) = b \wedge f(a) = c \wedge b \neq c$$

But $M[H] \vDash \mathsf{ZFC}$, giving a contradiction. $\square$

Theorem. If $\mathbb{P} \in M$ is a forcing poset and ($\mathbb{P}$ has the countable chain condition)^M, then $\mathbb{P}$ preserves cofinalities and hence cardinals.

Proof. Using the previous lemma, it suffices to show that $\mathbb{P}$ preserves regular cardinals. That is, if $\omega < \beta < \mathrm{Ord} \cap M$ and β is a limit, then if β is a regular cardinal in M, then β is a regular cardinal in $M[G]$. Suppose this is not the case, so there is such a β that is a regular cardinal in M but singular in $M[G]$. In $M[G]$, we can fix a cofinal map $f : \alpha \to \beta$ for some ordinal $\alpha < \beta$. As $\alpha, \beta \in M$, we can use the approximation lemma to find a function $F : \alpha \to \mathcal{P}^M(\beta)$ in M such that for all $\gamma \in \alpha$, we have $f(\gamma) \in F(\gamma)$ and $|F(\gamma)| \leq \aleph_0$. Working in M, let $X = \bigcup_{\gamma<\alpha} F(\gamma)$. This is a union of countable sets indexed by $\alpha < \beta$. So $X \subseteq \beta$ and is a subset of less than β-many countable sets. Hence $X \neq \beta$ as β is a regular cardinal in M. But f was cofinal, so $\beta = \bigcup_{\gamma<\alpha} f(\gamma) \subseteq X$, giving a contradiction. $\square$

4.4 The failure of the continuum hypothesis

Theorem. Let $\alpha < \mathrm{Ord} \cap M$, and let $\kappa = (\aleph_\alpha)^M$. Let $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$, and let G be $\mathbb{P}$-generic over M. Then $M[G]$ contains a κ-length sequence of distinct elements of 2^ω. Hence, $M[G] \vDash \mathsf{ZFC} + (\aleph_\alpha = \kappa \leq 2^{\aleph_0})$.

Proof. Let $f = \bigcup G \in M[G]$. Then f is a function $\kappa \times \omega \to 2$. For $\beta < \kappa$, let $g_\beta : \omega \to 2$ be the function given by $g_\beta(n) = f(\beta, n)$. We claim that for $\alpha \neq \beta$, we have $g_\alpha \neq g_\beta$. Define a dense set $E_{\alpha,\beta} \in M$ as follows.

$$E_{\alpha,\beta} = \{q \in \mathbb{P} \mid \exists n.\, \langle\beta, n\rangle, \langle\alpha, n\rangle \in \operatorname{dom} q \wedge q(\langle\beta, n\rangle) \neq q(\langle\alpha, n\rangle)\}$$

To show this is dense, fix $p \in \mathbb{P}$. Since p is finite, there is some m such that $\langle\beta, m\rangle, \langle\alpha, m\rangle \notin \operatorname{dom} p$. Define $q \leq p$ with $q : \operatorname{dom} p \cup \{\langle\beta, m\rangle, \langle\alpha, m\rangle\} \to 2$ by

$$q(z) = \begin{cases} p(z) & \text{if } z \in \operatorname{dom} p \\ 1 & \text{if } z = \langle\beta, m\rangle \\ 0 & \text{if } z = \langle\alpha, m\rangle \end{cases}$$

Since G is $\mathbb{P}$-generic, we can fix $q' \in G \cap E_{\alpha,\beta}$. Then

$$g_\beta(m) = f(\beta, m) = q(\langle\beta, m\rangle) \neq q(\langle\alpha, m\rangle) = f(\alpha, m) = g_\alpha(m)$$

Hence $g_\alpha \neq g_\beta$. Finally, since $\mathbb{P}$ has the countable chain condition in M, it preserves cardinals, so it preserves the $\aleph$ hierarchy. $\square$

In particular, if $\alpha = 2$, the model $M[G]$ satisfies $\neg\mathsf{CH}$.

Theorem. If ZFC is consistent, then so is ZFC + ¬CH.

The proof proceeds in the same way as the independence of V = L.

Definition. The g_β defined above are called *Cohen reals*. More precisely, we say that $c : \omega \to 2$ is a Cohen real over M if there exists H which is $\mathrm{Fn}(\omega, 2)$-generic over M and $c = \bigcup H$.

4.5 Possible sizes of the continuum

We have a way to add Cohen reals into a model M, but in general this process will add many more reals. In this subsection, we determine the possible sizes that the continuum can be. Recall that by König's theorem, $2^{\aleph_0} \neq \kappa$ for any κ with cofinality $\aleph_0$. We will show that this is the only restriction on the possible sizes of the continuum. Note that under GCH, for any κ, $\mathrm{cf}(\kappa) \neq \omega$ if and only if $\kappa^\omega = \kappa$.

Recall that in our proof that the axiom of power set holds in $M[G]$, given a name $\dot{a} \in M^{\mathbb{P}}$, the set $\mathcal{P}(\mathbb{P} \times \operatorname{ran} \dot{a})$ is a name for its power set. We will show that there is a better name that gives a tighter bound on the sizes of power sets.

Theorem. Let M be a transitive model of ZFC, and assume $(\kappa = \aleph_\alpha \wedge \kappa^\omega = \kappa)^M$. Let $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$, and let G be $\mathbb{P}$-generic over M. Then $M[G] \vDash 2^{\aleph_0} = \aleph_\alpha = \kappa$.

Proof. We have already shown that $M[G] \vDash$ ZFC and $M[G] \vDash \kappa = \aleph_\alpha \leq 2^{\aleph_0}$; it therefore remains to show that $2^{\aleph_0} \leq \aleph_\alpha$. Let $\dot{x}$ be a name for a subset of ω. For $n \in \omega$, let

$$E_{\dot{x},n} = \{p \in \mathbb{P} \mid (p \Vdash \check{n} \in \dot{x}) \vee (p \Vdash \check{n} \notin \dot{x})\}$$

This is dense in $\mathbb{P}$. For each $n \in \omega$, choose a maximal antichain $A_{\dot{x},n} \subseteq E_{\dot{x},n}$. This is shown to be possible on an example sheet using the axiom of choice. Define

$$\dot{z}_{\dot{x}} = \bigcup_{n \in \omega} \{\langle p, \check{n} \rangle \mid p \in A_{\dot{x},n} \wedge p \Vdash \check{n} \in \dot{x}\}$$

Such names are called *nice*. We will show that $\dot{z}_{\dot{x}}$ and $\dot{x}$ are both names for the same subset of ω, and since we can produce a bound on the amount of nice names, we can bound the size of $2^{\aleph_0}$.

We claim that $\mathbb{1} \Vdash \dot{x} = \dot{z}_{\dot{x}}$. To do this, it suffices to prove that for all $n \in \omega$,

$$D_{\dot{x},n} = \left\{q \in E_{\dot{x},n} \mid (q \Vdash \check{n} \in \dot{x}) \leftrightarrow (q \Vdash \check{n} \in \dot{z}_{\dot{x}})\right\}$$

is dense. Fix $n \in \omega$ and $p \in \mathbb{P}$. Since $E_{\dot{x},n}$ is dense, we can fix $p_0 \leq p$ such that $p_0 \in E_{\dot{x},n}$. As $A_{\dot{x},n}$ is a maximal antichain, there is $q_0 \in A_{\dot{x},n}$ such that $p_0 \parallel q_0$. Fix $r \leq p_0, q_0$. We will prove that $r \in D_{\dot{x},n}$. If $r \Vdash \check{n} \in \dot{x}$, then $q_0 \Vdash \check{n} \in \dot{x}$ as $q_0 \in E_{\dot{x},n}$. Hence, $\langle q_0, \check{n} \rangle \in \dot{z}_{\dot{x}}$ by definition, so $r \Vdash \check{n} \in \dot{z}_{\dot{x}}$. For the converse, suppose $r \Vdash \check{n} \in \dot{z}_{\dot{x}}$. By definition,

$$\{s \leq r \mid \exists \langle q_1, \check{m} \rangle \in \dot{z}_{\dot{x}}.\, s \leq q_1 \wedge (s \Vdash \check{m} = \check{n})\}$$

is dense below r. This can only happen if there is some q_1 with $\langle q_1, \check{n} \rangle \in \dot{z}_{\dot{x}}$ such that $r \parallel q_1$. Therefore, by definition, $q_1 \in A_{\dot{x},n}$. Since $A_{\dot{x},n}$ is an antichain containing q_0 and q_1 which are both compatible with r, we must have $q_0 = q_1$. Hence, $\langle q_0, \check{n} \rangle \in \dot{x}_{\dot{x}}$. Thus $q_0 \Vdash \check{n} \in \dot{x}$ by definition, so since $r \leq q_0$, we have $r \Vdash \check{n} \in \dot{x}$. Therefore $D_{\dot{x},n}$ is dense as required.

The total number of subsets of ω is therefore bounded by the number of nice names. First, note that $|\mathbb{P}| = \kappa$. Furthermore, since $\mathbb{P}$ has the countable chain condition, each $A_{\dot{x},n}$ is countable. Therefore, the amount of nice names is bounded by $(\kappa^\omega)^\omega \times (2^\omega)^\omega = \kappa$. As every subset of ω has a nice name, $M[G] \vDash 2^{\aleph_0} \leq \kappa$. □

Corollary. Con(ZFC) implies Con(ZFC + $(2^{\aleph_0} = \aleph_2)$), and (for example) Con(ZFC + $(2^{\aleph_0} = \aleph_{\omega_1})$).

Corollary. The following are equiconsistent.
(i) ZFC + there exists a weakly inaccessible cardinal;
(ii) ZFC + GCH + there exists a strongly inaccessible cardinal;
(iii) ZFC + $2^{\aleph_0}$ is weakly inaccessible;
(iv) ZFC + there exists a cardinal that is weakly inaccessible but not strongly inaccessible.

Proof. To show (i) implies (ii) we move to L. To show (iii) implies (iv), we note that $2^{\aleph_0}$ is not strongly inaccessible. It is trivial that (iv) implies (i). It therefore suffices to show that the continuum can be weakly inaccessible given (ii), which follows by considering the forcing $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$. □

Remark. When building models of ZFC + $(2^{\aleph_0} = \kappa)$, we often assume GCH for convenience. This can normally be done without loss of generality because we are usually only concerned with consistency results.

Example. Consider $\mathbb{P} = \mathrm{Fn}(\aleph_\omega^M \times \omega, 2)$. Let G be a $\mathbb{P}$-generic filter. Then in $M[G]$, we must have $2^{\aleph_0} \geq \aleph_\omega$. By König's theorem, this inequality must be strict. For convenience, assume GCH holds. Under this assumption, if $\mathrm{cf}(\kappa) = \omega$, then $\kappa^\omega = \kappa^+$, so there must be at most κ^+-many nice names. Hence $M[G] \vDash \aleph_\omega < 2^{\aleph_0} \leq \aleph_\omega^+$ which gives $M[G] \vDash 2^{\aleph_0} = \aleph_{\omega+1}$.

Remark. (i) Note that it is possible that $2^{\aleph_0} < \aleph_\omega$ but $\aleph_\omega^{\aleph_0} = \aleph_{\omega+1}^{\aleph_0} = \aleph_{\omega+2}$ without GCH. This can be proven using large cardinals.

(ii) If $M \vdash 2^{\aleph_0} = \aleph_\alpha > \aleph_\beta$ and $\mathbb{P} = \mathrm{Fn}(\aleph_\beta^M \times \omega, 2)$, then $M[G] \vDash 2^{\aleph_0} = \aleph_\alpha$.

(iii) The following are equiconsistent.

(a) ZFC + there exists a measurable cardinal + CH;

(b) ZFC + there exists a measurable cardinal + $\neg$CH.

The same holds for other large cardinal axioms such as huge cardinals and $I0$ to $I3$. We may also replace CH with GCH and the same holds.

(iv) The *proper forcing axiom*, which is a combinatorial axiom about forcing posets, implies that $2^{\aleph_0} = \aleph_2$ under ZFC.

4.6 Larger chain conditions

We now discuss generalised Cohen forcing. Suppose that we want a model of $\mathsf{ZFC} + \mathsf{CH} + (2^{\aleph_1} = \aleph_3)$. Naively, we might consider the forcing poset $\mathrm{Fn}(\omega_3 \times \omega_1, 2)$, but we can show that CH fails in this model.

Proposition. Let M be a countable transitive model of $\mathsf{ZFC} + \mathsf{GCH}$, and let $(\kappa = \aleph_\alpha \wedge \kappa^\omega = \kappa)^M$. Let $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$. Then, for any cardinal λ in M such that $\aleph_0 \leq \lambda < \kappa$, then in $M[G]$ we have

$$2^\lambda = \begin{cases} \kappa & \text{if } \operatorname{cf} \kappa > \lambda \\ \kappa^+ & \text{if } \operatorname{cf} \kappa \leq \lambda \end{cases}$$

There is a natural bijection between $\omega_3 \times \omega$ and $\omega_3 \times \omega_1$, and from this it will follow that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_3$.

Definition. Let I, J be sets and let κ be a regular cardinal. Define $\mathrm{Fn}_\kappa(I, J)$ to be the partial functions $I \to J$ of size less than κ. Its maximal element is $\varnothing$ under the order $q \leq p$ if and only if $p \subseteq q$.

Remark. (i) $\mathrm{Fn}_\omega(I, J) = \mathrm{Fn}(I, J)$.

(ii) The reason that $\mathrm{Fn}(I, J)$ was absolute is that finite objects are absolute. In general, $\mathrm{Fn}_\kappa(I, J)$ is not absolute. Moreover, if M is a countable transitive model, then $\mathrm{Fn}_\kappa(I, J) \notin M$. We instead need to consider the relativisation $(\mathrm{Fn}_\kappa(I, J))^M$.

(iii) If $\kappa > \omega$ and $I, J \neq \varnothing$, $\mathrm{Fn}_\kappa(I, J)$ does not have the countable chain condition.

(iv) If G is $\mathrm{Fn}_\kappa(I, J)$-generic over M, then $f = \bigcup G$ is a function $I \to J$.

Let $\mathbb{P} = \mathrm{Fn}_\kappa(\lambda \times \kappa, 2)$ where $\lambda \geq \kappa$ and κ is regular. Suppose also that $\lambda^\kappa = \lambda$. By a similar argument to the ω case, if $f = \bigcup G$ and $h_\alpha : \kappa \to 2$ is defined by $h_\alpha(\beta) = f(\alpha, \beta)$, then this gives a sequence of λ-many distinct functions $\kappa \to 2$. Similarly, by the nice names argument, there are precisely λ-many functions $\kappa \to 2$ because $\lambda^\kappa = \lambda$. We need to explicitly check that we have preserved all cardinals, using a generalisation of the countable chain condition. Once we have shown this, we will obtain $M[G] \vDash 2^\kappa = \lambda$.

Definition. For a cardinal κ, we say that $\mathbb{P}$ has the *κ-chain condition* if every antichain has cardinality less than κ.

The countable chain condition is equivalent to the $\aleph_1$-chain condition. All of the proofs above immediately generalise to the κ-chain condition.

Definition. We say that $\mathbb{P}$ *preserves cofinalities above κ* if and only if for all $\mathbb{P}$-generic filters G and limit ordinals $\gamma \in \mathrm{Ord} \cap M$ with $\operatorname{cf}^M(\gamma) \geq \kappa$, we have $\operatorname{cf}^M(\gamma) = \operatorname{cf}^{M[G]}(\gamma)$.

Lemma. Let $\mathbb{P} \in M$ be a forcing poset and $(\kappa \text{ is regular})^M$. Then

(i) $\mathbb{P}$ preserves cofinalities above κ if and only if for all $\mathbb{P}$-generic filters G and all limit ordinals β with $\kappa \leq \beta \in \mathrm{Ord} \cap M$, we have $(\beta \text{ is regular})^M \to (\beta \text{ is regular})^{M[G]}$;

(ii) If $\mathbb{P}$ preserves cofinalities above κ, then $\mathbb{P}$ preserves cardinals above κ.

Lemma. Let $A, B, \mathbb{P} \in M$, let $(\kappa \text{ is regular})^M$, let $(\mathbb{P} \text{ has the } \kappa\text{-chain condition})^M$, and let G be a $\mathbb{P}$-generic filter over M. Then for any $f : A \to B$ in $M[G]$, there is $F : A \to \mathcal{P}(B)$ in M such that for all $a \in A$, we have $f(a) \in F(a)$ and $(|F(a)| < \kappa)^M$.

Theorem. Let $\mathbb{P} \in M$ be a forcing poset such that $(\kappa \text{ is regular})^M$ and $(\mathbb{P} \text{ has the } \kappa\text{-chain condition})^M$. Then $\mathbb{P}$ preserves cofinalities above κ, and hence cardinals above κ.

On the example sheet, we show that for any infinite cardinal κ, $\mathrm{Fn}_\kappa(I, J)$ has the $(|J|^{<\kappa})^+$-chain condition. In particular, $\mathrm{Fn}_\kappa(\lambda \times \kappa, 2)$ has the $(2^{<\kappa})^+$-chain condition. We will show a different version of this theorem.

Lemma. Let κ be a regular cardinal in M, and suppose that $(2^{<\kappa} = \kappa)^M$. Then, if $(1 \leq |J| \leq 2^{<\kappa})^M$, the forcing poset $\mathbb{P} = \mathrm{Fn}_\kappa(I, J)^M$ has the κ^+-chain condition.

Proof. If I is empty, the result is trivial, so we may assume I is nonempty. Let W be an antichain in $\mathbb{P}$. To show that $|W| \leq \kappa$, we will construct chains $(A_\alpha)_{\alpha<\kappa}$ in I and $(W_\alpha)_{\alpha\in\kappa}$ such that

(i) for all $\alpha < \beta < \kappa$, we have $A_\alpha \subseteq A_\beta \subseteq I$ and $W_\alpha \subseteq W_\beta \subseteq W$;

(ii) for limit ordinals γ, we have $A_\gamma = \bigcup_{\beta<\gamma} A_\beta$ and $W_\gamma = \bigcup_{\beta<\gamma} W_\beta$;

(iii) $W = \bigcup_{\alpha<\kappa} W_\alpha$;

(iv) for all $\alpha < \kappa$, $|A_\alpha| \leq \kappa$ and $|W_\alpha| \leq \kappa$.

The result then follows by regularity of κ^+. Set $A_0 = W_0 = \varnothing$. It remains to define successor cases. Suppose we have constructed A_α, W_α. For each $p \in \mathbb{P}$ with $\mathrm{dom}\, p \subseteq A_\alpha$, using the axiom of choice we choose $q_p \in W$ such that $p = q_p|_{A_\alpha}$, if it exists. Note that if $\mathrm{dom}\, p \subseteq A_\beta$ for any $\beta < \alpha$, we will choose q_p to coincide with the q_p chosen at stage β. Then define

$$W_{\alpha+1} = W_\alpha \cup \{q_p \mid \mathrm{dom}\, p \subseteq A_\alpha\}$$

and

$$A_{\alpha+1} = \bigcup \{\mathrm{dom}\, q \mid q \in W_{\alpha+1}\}$$

Finally, set $A = \bigcup_{\alpha<\kappa} A_\alpha$.

We claim that $W = \bigcup_{\alpha<\kappa} W_\alpha$. By construction, we have $\bigcup_{\alpha<\kappa} W_\alpha \subseteq W$. For any $q \in W$, note that $\mathrm{dom}\, q \cap A \neq \varnothing$, otherwise take $q_1 \in W_1$, and $\mathrm{dom}\, q_1 \subseteq A$, so if $\mathrm{dom}\, q_1 \cap \mathrm{dom}\, q = \varnothing$, then $q_1 \parallel q$, contradicting $q_1, q \in W$. Since $\mathrm{dom}\, q \cap A = \varnothing$ and $|\mathrm{dom}\, q| < \kappa$, we must have $\mathrm{dom}\, q \cap A = \mathrm{dom}\, q \cap A_\alpha$ for some $\alpha < \kappa$. Define $p = q|_{A_\alpha}$. By definition, there is some $q' \in W_{\alpha+1}$ such that $q'|_{A_\alpha} = p$. Since $\mathrm{dom}\, q' \subseteq A$, we have $q \parallel q'$. As W is an antichain, this is only possible if $q = q'$, so $q \in \bigcup_{\alpha<\kappa} W_\alpha$.

We now show that for all $\alpha < \kappa$, the sets W_α and A_α have size at most κ. We show this by induction on α. The result for limit cases follows from regularity. If $|W_{\alpha+1}| \leq \kappa$, then clearly $|A_{\alpha+1}| \leq \kappa$, so it remains to show $|W_{\alpha+1}| \leq \kappa$. Since every condition q that is added to W_α is chosen from some condition p with $\operatorname{dom} p \subseteq A_\alpha$, then

$$|W_{\alpha+1}| \leq |W_\alpha| + |\{p \in \mathbb{P} \mid \operatorname{dom} p \subseteq A_\alpha\}|$$

As $|A_\alpha| \leq \kappa$ and $|\operatorname{dom} p| < \kappa$, then

$$|[A_\alpha]^{<\kappa}| \leq \kappa^{<\kappa} = 2^{<\kappa} = \kappa$$

Hence $|W_{\alpha+1}| \leq \kappa$ as required. □

Hence, if $\mathbb{P} = \operatorname{Fn}_\kappa(\lambda \times \kappa, 2)$, then $M[G] \vDash 2^\kappa = \lambda$ and all cardinals at least κ^+ are preserved.

4.7 Closure and distributivity

Definition. A poset $\mathbb{P}$ is $< \kappa$-*closed* if for every $\delta < \kappa$, every decreasing sequence of length δ in $\mathbb{P}$ has a lower bound.

Definition. $\mathbb{P}$ is $< \kappa$-*distributive* if the intersection of less than κ-many open dense sets is an open dense set.

Lemma. If $\mathbb{P}$ is $< \kappa$-closed then $\mathbb{P}$ is $< \kappa$-distributive.

Lemma. If κ is regular in M, then $\operatorname{Fn}_\kappa(I, J)^M$ is $< \kappa$-closed.

Theorem. Let $A, B, \mathbb{P} \in M$, let κ be a cardinal in M with $(|A| < \kappa)^M$, and suppose $\mathbb{P}$ is $< \kappa$-distributive in M. Let G be $\mathbb{P}$-generic. Then if $f \in M[G]$ with $f : A \to B$, then $f \in M$.

Informally, forcing over a distributive poset cannot add any new small functions.

Proof. It suffices to prove the statement for $A = \delta$ where $\delta < \kappa$. Suppose that $M[G] \vDash f : \delta \to B$. By the forcing theorem, there is $p \in G$ such that $p \Vdash \dot{f} : \check{\delta} \to \check{B}$. For $\alpha < \delta$, let

$$D_\alpha = \left\{q \leq p \mid \exists x \in B.\, q \Vdash \dot{f}(\check{\alpha}) = \check{x}\right\}$$

These sets are clearly open, and they are dense below p because p forces that $\dot{f}$ is a function. Since $\mathbb{P}$ is $< \kappa$-distributive, their intersection $D = \bigcap_{\alpha<\delta} D_\alpha$ is also (open and) dense below p. Let $q \in D \cap G$. Now, in M, for each $\alpha < \delta$, we can choose $x_\alpha \in B$ such that $q \Vdash \dot{f}(\check{\alpha}) = \check{x}_\alpha$, so we may define $g : \delta \to B$ by $\alpha \mapsto x_\alpha$. This g lies in M. But for any $\alpha < \delta$, we have $q \Vdash \dot{f}(\check{\alpha}) = \check{x}_\alpha = \check{g}(\check{\alpha})$, so $M[G] \vDash f = g$. □

Theorem. Let $I, J, \kappa \in M$. Suppose that κ is a regular cardinal in M, and $(2^{<\kappa} = \kappa \wedge |J| \leq \kappa)^M$. Then $\operatorname{Fn}_\kappa(I, J)^M$ preserves cofinalities and hence cardinals.

Proof. Recall that it suffices to show that for every limit ordinal $\beta \in \mathrm{Ord} \cap M$, if β is regular in M then β is regular in $M[G]$. Let β be regular in M.

Suppose that $\beta > \kappa$. Since $|J| \leq \kappa = 2^{<\kappa}$ in M, the forcing poset $\mathrm{Fn}_\kappa(I, J)^M$ has the κ^+-chain condition. So it preserves all cofinalities and cardinals at least κ^+, so in particular, β is regular in $M[G]$.

Now suppose that $\beta \leq \kappa$. Suppose that β is singular in $M[G]$. Fix $\delta < \beta$ and a cofinal map $f : \delta \to \beta$ in $M[G]$. Note that $\delta \in M$. Since $\mathbb{P}$ is $< \kappa$-closed, it is $< \kappa$-distributive, so $f \in M$, contradicting the assumption that β is regular in M. $\square$

Theorem. Let κ, λ be cardinals in M such that $\aleph_0 \leq \kappa \leq \lambda$. Suppose that κ is regular, $2^{<\kappa} = \kappa$, and $\lambda^\kappa = \lambda$ in M. Let $\mathbb{P} = \mathrm{Fn}_\kappa(\lambda \times \kappa, 2)$, and let G be $\mathbb{P}$-generic. Then $\mathbb{P}$ preserves cardinals, and $M[G] \vDash 2^\kappa = \lambda$.

We can use this to fix multiple sizes of power sets at once.

Theorem. Let M be a countable transitive model of ZFC + GCH. Then there is a countable transitive model of ZFC satisfying any of the following statements.

(i) $\mathsf{CH} + 2^{\aleph_1} = \aleph_3$;

(ii) $2^{\aleph_0} = 2^{\aleph_1} = \aleph_5$ and $2^{\aleph_2} = \aleph_{\omega+5}$;

(iii) for a fixed $n \in \omega$, for all $m \leq n$, $2^{\aleph_m} = \aleph_{2m+3}$.

Proof. Part (i). Let $\mathbb{P} = \mathrm{Fn}_{\aleph_1}(\omega_3 \times \omega_1, 2)^M$. If G is $\mathbb{P}$-generic, then $M[G] \vDash 2^{\aleph_1} = \aleph_3$. As $\mathbb{P}$ is ω_1-closed, it does not add any new functions $\omega \to 2$, so CH still holds in $M[G]$.

Part (ii). Let $\mathbb{P}_0 = \mathrm{Fn}_{\aleph_2}(\omega_{\omega+5} \times \omega_2, 2)^M$. Let G_0 be $\mathbb{P}_0$-generic. By closure, $2^{<\aleph_1} = \aleph_1$ in $M[G_0]$, and ${\aleph_5}^{\aleph_1} = \aleph_5$. Then let $\mathbb{P}_1 = \mathrm{Fn}_{\aleph_0}(\omega_5 \times \omega, 2)^{M[G_0]}$. Let G_1 be $\mathbb{P}_1$-generic. Then $M[G_1] \vDash 2^{\aleph_0} = 2^{\aleph_1} = \aleph_5$, where the latter equality is due to the fact that if M is a model of ZFC + GCH and G is $\mathrm{Fn}(\kappa \times \omega, 2)$-generic, then for any cardinal $\lambda \in M$ with $\aleph_0 \leq \lambda < \kappa$, the value of 2^λ in $M[G]$ is κ if $\mathrm{cf}(\kappa) > \lambda$ and κ^+ if $\mathrm{cf}(\kappa) \leq \lambda$. Also, $M[G_1] \vDash 2^{\aleph_2} = \aleph_{\omega+5}$ by preservation of cardinals.

Part (iii) is similar; we first make $2^{\aleph_m} = \aleph_{2m+3}$, then make $2^{\aleph_{m-1}} = \aleph_{2(m-1)+3}$, and continue downwards. $\square$

Remark. (i) It is necessary to start at the largest cardinal and work downwards; this ensures that the cardinal arithmetic in our forcing models remains correct.

(ii) The iterative approach works for any finite number of cardinals. We will see later how we can force $2^{\aleph_n} = \aleph_{2n+3}$ for all $n \in \omega$.

We give an example to show that the order described in (i) is necessary.

Proposition. Let M be a countable transitive model of ZFC with $M \vDash 2^{\aleph_0} = \aleph_\alpha$. Let $\mathbb{P} = \mathrm{Fn}_{\aleph_1}(\kappa \times \aleph_1, 2)$ for some $\kappa \geq 1$. Then if G is $\mathbb{P}$-generic, $M[G] \vDash \mathsf{CH}$, and all cardinals δ of M with $\aleph_1 \leq \delta \leq \aleph_\alpha$ in M are no longer cardinals in $M[G]$. In particular, $\aleph_\alpha^M \neq \aleph_\alpha^{M[G]}$.

This is on the example sheets.

4.8 The mixing lemma

Recall that $p \Vdash \exists x.\, \varphi(x)$ if and only if

$$\left\{q \leq p \mid \exists \dot{x} \in \mathrm{V}^{\mathbb{P}}.\, q \Vdash \varphi(\dot{x})\right\}$$

is dense below p. In most cases, the witness $\dot{x}$ does not depend on G. For example, in $p \Vdash \exists x.\, (\dot{a} \in x \wedge \dot{b} \in x)$, we can find a name $\dot{x} = \mathrm{op}(\dot{a}, \dot{b})$ without needing to know G. Informally, the mixing lemma says that this is always the case, as long as M has AC.

Theorem (the mixing lemma)**.** (ZFC) Suppose that $(p \Vdash \exists x.\, \varphi(x))^M$. Then there is a name $\dot{x} \in M^{\mathbb{P}}$ such that $(p \Vdash \varphi(\dot{x}))^M$.

Proof. Since

$$\left\{q \leq p \mid \exists \dot{x} \in M^{\mathbb{P}}.\, q \Vdash \varphi(\dot{x})\right\}$$

is dense below p, it contains a maximal antichain D. Now, for each $q \in D$, choose some $\dot{x}_q$ such that $q \Vdash \varphi(\dot{x}_q)$. Without loss of generality, we may assume that if $\langle r, \dot{y}\rangle \in \dot{x}_q$, then $r \leq q$. This is because

(i) if $r \perp q$, then $q \Vdash \dot{x}_q = (\dot{x}_q \setminus \langle r, \dot{y}\rangle)$; and

(ii) if $r \parallel q$, then define

$$\dot{x}'_q = (\dot{x}_q \setminus \langle r, \dot{y}\rangle) \cup \{\langle s, \dot{y}\rangle \mid s \leq r, q\}$$

so $q \Vdash \dot{x}_q = \dot{x}'_q$.

Now, if $q, q' \in D$ are such that $q \neq q'$, we must have $q \perp q'$ as D is an antichain. So $q' \Vdash \dot{x}_q = \varnothing$. We 'mix' the $\dot{x}_q$ together to form

$$\dot{x} = \bigcup \left\{\dot{x}_q \mid q \in D\right\}$$

Then if $q \in D$, we have $q \Vdash \dot{x} = \dot{x}_q$. By the forcing theorem, $q \Vdash \varphi(\dot{x})$.

It remains to show that $p \Vdash \varphi(\dot{x})$. Suppose otherwise, so there is $r \leq p$ such that $r \Vdash \neg\varphi(\dot{x})$. As D is a maximal antichain of conditions below p, there is a condition $q \in D$ such that $q \parallel r$. Now if $s \leq q, r$, we have $s \Vdash \varphi(\dot{x})$ and $s \Vdash \neg\varphi(\dot{x})$, giving a contradiction. □

4.9 Forcing successor cardinals

We would now like to find forcing posets that collapse $\kappa < \lambda$ such that $\lambda = \kappa^+$. Observe that this can only happen if λ is regular in M. This is because if $f : \alpha \to \lambda$ is cofinal with $\alpha < \lambda$ and $f \in M$, then $f \in M[G]$, so

$$\mathrm{cf}^{M[G]}(\lambda) \leq \mathrm{cf}^{M[G]}(\alpha) \leq |\alpha|^{M[G]} < \lambda$$

Assuming GCH in the ground model, this is the only restriction. We will prove this in the case where λ is a successor cardinal, and in the case where λ is strongly inaccessible; given GCH, these are the only options.

Theorem. Let κ be a regular cardinal in M, and let $\delta > \kappa$ be a cardinal in M. Let $\lambda = \delta^+$ in M. Let G be $\mathrm{Fn}_\kappa(\kappa, \delta)$-generic over M. Then in $M[G]$,

(i) $|\delta| = \kappa$;

(ii) every cardinal $\alpha \leq \kappa$ in M remains a cardinal in $M[G]$;

(iii) if $\delta^{<\kappa} = \delta$ then every cardinal $\alpha > \delta$ in M remains a cardinal in $M[G]$.

In particular, if $\delta^{<\kappa} = \delta$, then $M[G] \vDash \lambda = \kappa^+$.

Observe that if δ is a cardinal in M and $\delta > |\mathbb{P}|$ in M, then δ remains a cardinal in $M[G]$. This is because $\mathbb{P}$ has the $|\mathbb{P}|^+$-chain condition.

Proof. Part (i). Note that $\bigcup G : \kappa \to \delta$ is a surjection, so $|\delta| = |\kappa|$ in $M[G]$. In particular, there are no cardinals between δ and λ.

Part (ii). Since κ is regular, $\mathrm{Fn}_\kappa(\kappa, \delta)$ is $< \kappa$-closed, so every cardinal $\alpha \leq \kappa$ is preserved.

Part (iii). Finally, if $\delta^{<\kappa} = \delta$, then $|\mathrm{Fn}_\kappa(\kappa, \delta)| = \delta$, so $\mathrm{Fn}_\kappa(\kappa, \delta)$ has the δ^+-chain condition, so every cardinal $\alpha > \delta$ (in particular, λ) is preserved. □

We can force inaccessible cardinals λ to become successor cardinals. To do this, we will use a forcing poset called the *Lévy collapse*.

Definition. Let $\lambda > \kappa$ be infinite ordinals. Then $\mathrm{Col}(\kappa, < \lambda)$ consists of all functions p such that

(i) p is a partial function from $\kappa \times \lambda \to \lambda$;
(ii) $|\mathrm{dom}\, p| < \kappa$;
(iii) $p(\alpha, \beta) < \beta$ for each $(\alpha, \beta) \in \mathrm{dom}\, p$.

We make this into a forcing poset by writing $q \leq p$ if and only if q extends p as a function.

Informally, for each $\beta < \lambda$, we add a surjection $\kappa \to \beta$.

Theorem (Lévy)**.** Let κ be a regular cardinal in M, and suppose $\lambda > \kappa$ is strongly inaccessible in M. Let G be $\mathrm{Col}(\kappa, < \lambda)$-generic over M. Then in $M[G]$,

(i) every ordinal β with $\kappa \leq \beta < \lambda$ has cardinality κ; and
(ii) every cardinal at most κ or at least λ remains a cardinal.

In particular, $M[G] \vDash \lambda = \kappa^+$.

Proof. If $\beta < \lambda$, we can define $G_\beta : \kappa \to \beta$ by $G_\beta(\alpha) = \left(\bigcup G\right)(\alpha, \beta)$. By density, this is a surjection, so if $\kappa \leq \beta < \lambda$, we have $M[G] \vDash |\beta| = |\kappa|$.

Note that $\mathrm{Col}(\kappa, < \lambda)$ is $< \kappa$-closed, so preserves cardinals at most κ. In particular, κ remains a cardinal.

Now, $|\mathrm{Col}(\kappa, < \lambda)| = \lambda$. Therefore, $\mathrm{Col}(\kappa, < \lambda)$ has the λ^+-chain condition and therefore preserves cardinals at least λ^+.

Finally, we show that λ is still a cardinal in $M[G]$, which follows from the λ-chain condition. Given $p \in \mathrm{Col}(\kappa, < \lambda)$, define the *support* of p to be

$$\mathrm{sp}(p) = \{\beta \mid \exists \alpha. \langle \alpha, \beta \rangle \in \mathrm{dom}\, p\}$$

As $|p| < \kappa$, we must have $|\mathrm{sp}(p)| < \kappa$. Let W be an antichain. We will construct chains $(A_\alpha)_{\alpha<\kappa}$ and $(W_\alpha)_{\alpha<\kappa}$ such that

(i) for $\alpha < \beta < \kappa$, $A_\alpha \subseteq A_\beta \subseteq \lambda$ and $W_\alpha \subseteq W_\beta \subseteq W$;
(ii) if $\gamma < \kappa$ is a limit, then $A_\gamma = \bigcup_{\alpha<\gamma} A_\alpha$ and $W_\gamma = \bigcup_{\alpha<\gamma} W_\alpha$;

(iii) $W = \bigcup_{\alpha<\kappa} W_\alpha$;

(iv) for all $\alpha < \kappa$, $|A_\alpha|, |W_\alpha| < \lambda$.

Assuming this can be done, since λ is regular, we have $|W| = \left|\bigcup_{\alpha<\kappa} W_\alpha\right| < \lambda$. To do this, first set $A_0 = W_0 = \varnothing$. To define successor cases, suppose A_α, W_α are defined. Suppose that $p \in \mathrm{Col}(\kappa, <\lambda)$ has $\mathrm{sp}(p) \subseteq A_\alpha$. Using the axiom of choice, choose $q_p \in W$ such that $p = q_p|_{\kappa\times\mathrm{sp}(p)}$ if this exists. Define

$$W_{\alpha+1} = \{q_p \mid \mathrm{sp}(p) \subseteq A_\alpha\}; \quad A_{\alpha+1} = \bigcup\{\mathrm{sp}(q) \mid q \in W_{\alpha+1}\}$$

One can show that $W = \bigcup_{\alpha<\kappa} W_\alpha$ in the same way that we proved this for $\mathrm{Fn}_\kappa(I, J)$. We show by induction that for $\alpha < \kappa$, $|A_\alpha|, |W_\alpha| < \lambda$. Limit cases follow by regularity. If $|W_{\alpha+1}| < \lambda$, then $|A_{\alpha+1}| < \kappa \cdot \lambda = \lambda$. Suppose $|A_\alpha| < \lambda$. Then, since every q added in stage $\alpha + 1$ is chosen from some condition with support contained in A_α, we must have

$$|W_{\alpha+1}| \leq |A_\alpha|^{<\kappa}$$

Then as λ is a strong limit, $|A_\alpha|^{<\kappa} < \lambda$. □

Remark. (i) The requirement that κ was regular allowed us to deduce κ-closure.

(ii) Suppose λ is weakly inaccessible and $2^{\aleph_0} > \lambda$. Then $\mathrm{Col}(\aleph_1, <\lambda)$ has an antichain of length $2^{\aleph_0}$, so will not satisfy the λ-chain condition. Indeed, for $A \subseteq \omega$, we define $p_A : \{\omega\}\times[\omega, \omega+\omega) \to 2$ by

$$p_A(\alpha, \omega + n) = \begin{cases} 0 & \text{if } n \in A \\ 1 & \text{if } n \notin A \end{cases}$$

Then if $A \neq B$, the functions p_A, p_B are incompatible.

(iii) One can show that λ is weakly compact if and only if it is inaccessible and satisfies the *tree property*. We claim that if G is $\mathrm{Col}(\aleph_0, <\lambda)$-generic, then in $M[G]$, $\aleph_1$ has the tree property. In general, we can use forcing to add combinatorial properties from large cardinals to $\aleph_1$.

(iv) This shows that λ being a limit cardinal is not absolute between M and N, even if λ being a cardinal is absolute for M, N.

Corollary. If ZFC + IC is consistent, then so is ZFC + ($\aleph_1^{\mathrm{V}}$ is inaccessible in L).

Proof. Start with a model of V = L where λ is inaccessible, and let G be $\mathrm{Col}(\omega_1, <\lambda)$-generic. Then $M[G] \vDash \lambda = \aleph_1$, but also $M[G] \vDash (\lambda \text{ is inaccessible})^{\mathrm{L}}$. □

Remark. If V $\vDash$ ZFC + κ is measurable, then for example, $\aleph_1^{\mathrm{V}}$ is inaccessible in L.

4.10 Product forcing

In this subsection, we will show that is consistent that, for example, each $n \in \omega$ satisfies $2^{\aleph_n} = \aleph_{2n+3}$. We have already shown that for a fixed $N \in \omega$, it is consistent that all $n < N$ have $2^{\aleph_n} = \aleph_{2n+3}$. However, we cannot get this result using the iterated forcing process described in previous sections, and will instead use *product forcing*. This technique will allow us to exactly determine the restrictions on the continuum function F : Card → Card given by $F(\aleph_\alpha) = 2^{\aleph_\alpha}$.

Definition. Suppose $(\mathbb{P}, \leq_{\mathbb{P}})$ and $(\mathbb{Q}, \leq_{\mathbb{Q}})$ are posets. The *product order* $\leq$ on $\mathbb{P}\times\mathbb{Q}$ is defined by
$$\langle p_1, q_1\rangle \leq \langle p_0, q_0\rangle \leftrightarrow p_1 \leq_{\mathbb{P}} p_0 \wedge q_1 \leq_{\mathbb{Q}} q_0$$

Given a $\mathbb{P} \times \mathbb{Q}$-generic filter G over M, we can produce the *projections*

$$G_0 = \{p \in \mathbb{P} \mid \exists q \in \mathbb{Q}. \langle p, q\rangle \in G\}$$
$$G_1 = \{q \in \mathbb{Q} \mid \exists p \in \mathbb{P}. \langle p, q\rangle \in G\}$$

Lemma. Let M be a transitive model of ZFC with $\mathbb{P}, \mathbb{Q} \in M$. Let $G \subseteq \mathbb{P}$ and $H \subseteq \mathbb{Q}$. Then the following are equivalent.

(i) $G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic over M;
(ii) G is $\mathbb{P}$-generic over M and H is $\mathbb{Q}$-generic over $M[G]$;
(iii) H is $\mathbb{Q}$-generic over M and G is $\mathbb{P}$-generic over $M[H]$.

Moreover, when this is the case, $M[G \times H] = M[G][H] = M[H][G]$.

Proof. The first part is left as an exercise. For the last part, recall that the generic model theorem shows that if N is a transitive model of ZF containing M as a definable class and containing G as a set, then $M[G] \subseteq N$. Since $M \subseteq M[G][H]$, and $G \times H$ is an element of $M[G][H]$, we obtain $M[G \times H] \subseteq M[G][H]$. For the other direction, $G \in M[G \times H]$ and $M \subseteq M[G \times H]$ so $M[G] \subseteq M[G \times H]$, but also $H \in M[G \times H]$ so $M[G][H] \subseteq M[G \times H]$. □

Recall that we started with a model of ZFC + GCH and forced with

$$G_0 \text{ is } \mathrm{Fn}(\omega_3 \times \omega, 2)^M\text{-generic}; \quad G_1 \text{ is } \mathrm{Fn}(\omega_5 \times \omega_1, 2)^{M[G_0]}\text{-generic}$$

and found that $M[G_0][G_1] \vDash \mathsf{CH}$. But if instead we used

$$G_0 \text{ is } \mathbb{P}_0 = \mathrm{Fn}(\omega_5 \times \omega_1, 2)^M\text{-generic}; \quad G_1 \text{ is } \mathbb{P}_1 = \mathrm{Fn}(\omega_3 \times \omega, 2)^{M[G_0]}\text{-generic}$$

then we obtain $M[G_0][G_1] \vDash 2^{\aleph_0} = \aleph_3 + 2^{\aleph_1} = \aleph_5$. However, $\mathbb{P}_0$ is $< \omega_1$-closed, so does not add new sequences of length ω. Thus $\mathbb{P}_1 = \mathrm{Fn}(\omega_3 \times \omega, 2)^M$. We can therefore define the forcing poset $\mathbb{P}_0 \times \mathbb{P}_1$-over M, and $G_0 \times G_1$ is $\mathbb{P}_0 \times \mathbb{P}_1$-generic over M. To simultaneously force $2^{\aleph_n} = \aleph_{2n+3}$, we use the poset
$$\mathbb{P} = \prod_{n\in\omega} \mathrm{Fn}_{\omega_n}(\omega_{2n+3} \times \omega_n, 2)$$

Easton's theorem shows that this works.

Theorem (Easton's theorem for sets)**.** Let M be a countable transitive model of ZFC + GCH. Let S be a set of regular cardinals in M, and let $F : S \to \mathrm{Card}^M$ be a function in M such that for all $\kappa \leq \lambda$ in S,

(i) $F(\kappa) > \kappa$ (Cantor's theorem);
(ii) $F(\kappa) \leq F(\lambda)$ (monotonicity);
(iii) $\mathrm{cf}(F(\kappa)) > \kappa$ (König's theorem).

Then there is a generic extension $M[G]$ of M such that $M, M[G]$ have the same cardinals, and for all $\kappa \in S$, $M[G] \vDash 2^\kappa = F(\kappa)$.

The proof is non-examinable.

By essentially the same proof, this result can be generalised to proper classes of M, and in particular $S = \mathrm{Reg}^M$. This needs a notion of *class forcing*, as $\mathbb{P}$ is a proper class. The main obstacle with class forcing is that $M[G]$ need not be a model of ZFC. For example, consider $\mathrm{Fn}(\mathrm{Ord}\times\omega, 2)$, which makes $2^{\aleph_0}$ a proper class. Alternatively, consider $\mathrm{Fn}(\omega, \mathrm{Ord})$, which creates a surjection $\bigcup G : \omega \to \mathrm{Ord}$. In fact, the forcing relation $\Vdash$ may not even be definable. However, one can show that the particular forcing poset used in Easton's theorem also satisfies all of the required results for the proofs to work. In conclusion, we can say almost nothing about the values of the continuum function.