Algebraic Geometry

Cambridge University Mathematical Tripos: Part II

17th May 2024

Contents

1 Affine varieties

1.1 Introduction

Algebraic geometry studies the duality between systems of polynomial equations and the geometry or topology of their solution sets. If we have a system of polynomials

$$f_1, \dots, f_r \in \Bbbk[X_1, \dots, X_n] = \Bbbk[\mathbf{X}]$$

we can form its solution set

$$V = \{P \in \Bbbk^n \mid f_1(P) = \dots = f_r(P) = 0\} \subseteq \Bbbk^n$$

On the algebraic side, we have the ideal

$$I = (f_1, \dots, f_r) \lhd \Bbbk[\mathbf{X}]$$

The duality we are interested in is between $R = {}^{\Bbbk[\mathbf{X}]}\!/_{I}$ and the geometry of V.

We may impose some assumptions on the field $\Bbbk$.

- We might assume that $\Bbbk$ is algebraically closed, which is a natural assumption since we wish to consider roots to polynomials with coefficients in $\Bbbk$.
- We could also take the stronger assumption that $\Bbbk$ is algebraically closed and has characteristic 0. Occasionally, we may want to differentiate a polynomial, and so it becomes inconvenient to do algebra without this assumption.
- Throughout the course, we will in fact assume $\Bbbk = \mathbb{C}$, as we are not particularly interested in the subtleties of such fields other than $\mathbb{C}$, and it is useful for intuition.

Questions we may ask about this duality are:

- To what extent do R and V determine each other?
- What is the right notion of dimension of V, in terms of algebra?
- Can we detect whether $V \subseteq \mathbb{C}^n$ is a manifold based on the information contained within R?
- Is V compact? If not, is there a natural way to compactify the space into some space $\overline{V}$ that is in some sense algebraic?

1.2 Affine space

Definition. The *affine space of dimension n*, implicitly over $\mathbb{C}$, is the set $\mathbb{A}^n = \mathbb{C}^n$. The elements of $\mathbb{A}^n$ are called *points*, denoted $P = (\mathbf{a}) = (a_1, \dots, a_n)$.

Definition. An *affine subspace* of $\mathbb{A}^n$ is any subset of the form $v + U \subseteq \mathbb{C}^n$ where $U \subseteq \mathbb{C}^n$ is any linear subspace, and $v \in \mathbb{C}^n$.

$\mathbb{A}^n$ is the natural set on which $\mathbb{C}[X_1, \dots, X_n]$ is a ring of functions. Given $f \in \mathbb{C}[\mathbf{X}]$, we obtain a function $f : \mathbb{A}^n \to \mathbb{C}$. The subset $\mathbb{C} \subseteq \mathbb{C}[\mathbf{X}]$ is the set of constant functions.

Proposition. The polynomial ring $\mathbb{C}[\mathbf{X}]$ satisfies the following properties.

(i) $\mathbb{C}[\mathbf{X}]$ is a unique factorisation domain.

(ii) Every ideal in $\mathbb{C}[\mathbf{X}]$ is finitely generated (equivalently, $\mathbb{C}[\mathbf{X}]$ is Noetherian), due to the Hilbert basis theorem.

1.3 Affine varieties

Definition. Let $S \subseteq \mathbb{C}[\mathbf{X}]$ be any subset of $\mathbb{C}[\mathbf{X}]$. The *vanishing locus* of S is defined to be $\mathbb{V}(S) = \{P \in \mathbb{A}^n \mid \forall f \in S,\ f(P) = 0\}$.

Definition. An *affine (algebraic) variety in* $\mathbb{A}^n$ is a set of the form $\mathbb{V}(S)$ for some S.

Note that there is some inconsistency between definitions in different textbooks; some authors also impose an irreducibility condition.

Example. (i) Let $n = 1$. The polynomial $f \in \mathbb{C}[X]$ gives the vanishing locus $\mathbb{V}(f) \subseteq \mathbb{A}^1$, the set of zeroes of f. Conversely, if $V \subseteq \mathbb{A}^1$ is finite, then $V = \mathbb{V}(f)$ where $f = \prod_{a \in V}(x - a)$.

(ii) A *hypersurface* in $\mathbb{A}^n$ is a variety of the form $\mathbb{V}(f)$ where $f \in \mathbb{C}[X]$.

(iii) It is often convenient to represent varieties not by equations but parametrically. The *affine twisted cubic* is $C = \left\{(t, t^2, t^3) \mid t \in \mathbb{C}\right\} \subset \mathbb{A}^3$. This is a variety, as it is the vanishing locus of the two polynomials $X_1^2 - X_2$ and $X_1^3 - X_3$.

Theorem. Let $S \subseteq \mathbb{C}[\mathbf{X}]$. Then,

(i) Let $I \subseteq \mathbb{C}[\mathbf{X}]$ be the ideal generated by S. Then, $\mathbb{V}(S) = \mathbb{V}(I)$.

(ii) There exists a finite subset $\{f_j\}$ of S such that $\mathbb{V}(S) = \mathbb{V}(\{f_j\})$.

Proof. Part (i). Suppose $P \in \mathbb{A}^n$. Then, $f(P) = 0$ for all $f \in S$ if and only if $f(P) = 0$ for all $f \in I$, by the basic properties of ideals.

Part (ii). By (i), $\mathbb{V}(S) = \mathbb{V}(I)$. I is finitely generated, so there exist functions $h_1, \dots, h_r \in I$ that generate I. Reversing (i), $\mathbb{V}(I) = \mathbb{V}(\{h_i\})$. But since I is generated by S, each h_i can be written as a linear combination of finitely many elements of S. So $h_i = \sum_j g_{ij} f_j$ where $f_j \in S$. Then $\mathbb{V}(S) = \mathbb{V}(\{f_j\})$. □

Proposition. Let $S, T \subseteq \mathbb{C}[\mathbf{X}]$. Then,

(i) $S \subseteq T$ implies $\mathbb{V}(T) \subseteq \mathbb{V}(S)$.

(ii) $\mathbb{V}(0) = \mathbb{A}^n$, and $\mathbb{V}(\mathbb{C}[\mathbf{X}]) = \mathbb{V}(\lambda) = \varnothing$ where $\lambda \in \mathbb{C} \setminus \{0\}$.

(iii) $\bigcap_j \mathbb{V}(I_j) = \mathbb{V}\left(\sum_j I_j\right)$ for any family of ideals I_j.

(iv) $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(I \cap J)$.

Proof. Part (i) and (ii) are trivial.

Part (iii). We have $\bigcap_j \mathbb{V}(I_j) = \mathbb{V}\left(\bigcup_j I_j\right)$. To conclude, note that the ideal generated by $\bigcup_j I_j$ is $\sum_j I_j$.

Part (iv). We have already seen that $\mathbb{V}(I) \cup \mathbb{V}(J) \subseteq \mathbb{V}(I \cap J)$. For the reverse containment, suppose $P \in \mathbb{V}(I \cap J)$, and suppose $P \notin \mathbb{V}(I)$. Then, there exists some $g \in I$ such that $g(P) = 0$. Moreover, for all elements $f \in J$, $fg \in I \cap J$, so $(fg)(P) = 0$. Hence $f(P) = 0$ for all $f \in J$, so $P \in \mathbb{V}(J)$. $\square$

1.4 Irreducible varieties

Definition. A variety V is called *irreducible* if whenever $V = V_1 \cup V_2$, where V_1, V_2 are varieties, we have $V = V_1$ or $V = V_2$. A variety that is not irreducible is called reducible.

Example. The variety $V = \mathbb{V}(XY)$ is reducible, as it is the union of $\mathbb{V}(X)$ and $\mathbb{V}(Y)$.

Proposition. Every affine variety V is a finite union of irreducible varieties.

This proof uses a 'bisection' argument.

Proof. If V is irreducible, there is nothing to prove. Otherwise, $V = V_1 \cup V_1'$, where $V_1, V_1' \neq V$. If V_1, V_1' are finite unions of irreducible varieties, the proof is already complete. Suppose V_1 is not a finite union of irreducibles. Then, it follows that $V_1 = V_2 \cup V_2'$ nontrivially. Inductively, we obtain

$$V = V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq V_3 \supsetneq \dots$$

This infinite descending chain never stabilises. Define

$$W = \bigcap_{j=0}^{\infty} V_j = \mathbb{V}\left(\sum_{j=0}^{\infty} I_j\right)$$

But $\sum_{j=0}^{\infty} I_j$ is finitely generated. So $\sum_{j=0}^{\infty} I_j = \sum_{j \leq N} I_j$ for some $N \in \mathbb{N}$. Hence, $W = \bigcap_{j \leq N} V_j$ contradicting that the descending chain never stabilises. $\square$

Definition. Let V be an affine variety. A *minimal decomposition* of V is a representation of V as a finite union of distinct irreducibles V_i such that no V_i is contained within V_j.

Proposition. Minimal decompositions of affine varieties are unique up to ordering.

Proof sketch. This proof is left as an exercise. One can compare two decompositions by intersecting the irreducible components of one decomposition with the other. $\square$

Given uniqueness of minimal decompositions, we can refer to the irreducibles appearing in such a decomposition as the *irreducible components* of a variety.

1.5 Zariski and Euclidean topologies

Definition. The *Zariski topology* on $\mathbb{A}^n$ is the topology where the closed sets are precisely the affine varieties. If $V \subseteq \mathbb{A}^n$ is a (sub)variety, the Zariski topology on V is the subspace topology for the Zariski topology on $\mathbb{A}^n$.

Remark. This is in fact a topology, as all of the relevant axioms have been proven.

Definition. The *Euclidean topology* or *analytic topology* on $\mathbb{A}^n$ is the topology induced by the metric space structure on $\mathbb{C}^n$. If $V \subseteq \mathbb{A}^n$, the Euclidean topology on V is the subspace topology of the Euclidean topology on $\mathbb{A}^n$.

Proposition. The Zariski topology on $\mathbb{A}^1$ coincides with the cofinite topology; the closed sets are exactly the finite sets. This topology is not Hausdorff but it is compact. The Euclidean topology on $\mathbb{A}^1$ is Hausdorff but not compact.

Remark. $\mathbb{A}^2$ with the Zariski topology is not homeomorphic to $\mathbb{A}^1 \times \mathbb{A}^1$ with the product of the Zariski topologies.

1.6 Ideals from zero sets

Theorem (weak form of Hilbert's Nullstellensatz)**.** Every maximal ideal in $\mathbb{C}[\mathbf{X}]$ has the form $(X_1 - a_1, \dots, X_n - a_n)$ for $a_i \in \mathbb{C}$. Moreover, if I is any non-unit ideal, $\mathbb{V}(I) \neq \varnothing \subseteq \mathbb{A}^n$.

We prove this over the complex numbers; the given proof only works for this case, but the statement holds for all algebraically closed fields.

Proof. Every ideal of this form has quotient $\mathbb{C}$, so they are all maximal. Let $\mathfrak{m} \lhd \mathbb{C}[\mathbf{X}]$ be a maximal ideal, and let $K = {}^{\mathbb{C}[\mathbf{X}]}\!/_{\mathfrak{m}}$. K is a field as $\mathfrak{m}$ is maximal, and it is a field extension of $\mathbb{C}$. Define a_i to be the coset $X_i + \mathfrak{m}$. If $a_i \in \mathbb{C}$ for all i, this gives the result as required because the ideal is generated by $(X_1 - a_1, \dots, X_n - a_n)$.

Otherwise, $K \supsetneq \mathbb{C}$. But $\mathbb{C}$ is algebraically closed, so there exists $t \in K \setminus \mathbb{C}$ which is transcendental over $\mathbb{C}$. Let U_m be the $\mathbb{C}$-span inside K of products of the form $a_1^{r_1} \dots a_n^{r_n}$ where the r_i are nonnegative, and $\sum_{i=1}^n r_i \leq m$. Observe that U_m is finite-dimensional, and $K = \bigcup_{m \geq 0} U_m$ is countable-dimensional. One can show that the elements $\left\{\frac{1}{t-c} \mid c \in \mathbb{C}\right\}$ are linearly independent over $\mathbb{C}$. There are uncountably many such elements, giving a contradiction.

For the last part, let I be a nonzero ideal. There exists a maximal ideal $\mathfrak{m} \supseteq I$, so $\mathbb{V}(I) \supseteq \mathbb{V}(\mathfrak{m})$, but $\mathbb{V}(\mathfrak{m})$ is nonempty as it contains the point $(a_1, \dots, a_m)$. $\square$

Definition. Let $V \subseteq \mathbb{A}^n$ be an affine variety. The *ideal of functions vanishing on V* is $I(V) = \{f \in \mathbb{C}[\mathbf{X}] \mid \forall P \in V,\ f(P) = 0\}$.

Proposition. Let $V \subseteq \mathbb{A}^n$ be an affine variety. Then,

(i) If $V = \mathbb{V}(S)$ where $S \subseteq \mathbb{C}[\mathbf{X}]$, then $S \subseteq I(V)$. In particular, $I(V)$ is the largest ideal vanishing on V.

(ii) $V = \mathbb{V}(I(V))$.

(iii) Varieties $V, W \subseteq \mathbb{A}^n$ are equal if and only if $I(V) = I(W)$.

Proof. Follows from the definitions. □

Therefore, we have an injective map I from the space of affine varieties in $\mathbb{A}^n$ to the space of ideals in $\mathbb{C}[\mathbf{X}]$, and $\mathbb{V}$ gives a left inverse.

Proposition. If V, W are affine varieties, $V \subseteq W$ if and only if $I(W) \subseteq I(V)$.

Proof. The forward implication follows from set theory. For the reverse, if $V \nsubseteq W$, we can choose $P \in V \setminus W$. Since $P \notin \mathbb{V}(I(W))$, there exists a function $f \in I(W)$ such that $f(P) \neq 0$, so $f \notin I(V)$. □

Proposition. Let V be a variety. Then V is irreducible if and only if $I(V)$ is a prime ideal.

Recall that $I(V)$ is prime when $f_1 f_2 \in I(V)$ implies $f_1 \in I(V)$ or $f_2 \in I(V)$. Geometrically, the ideal is not prime when we can find two functions where the product is zero on V but are individually not zero on all of V.

Proof. Recall that $I(V_1 \cup V_2) = I(V_1) \cap I(V_2)$. Suppose V were reducible, so $V = V_1 \cup V_2$ where $V_1, V_2 \neq V$. In particular, $V_1 \nsubseteq V_2 \nsubseteq V_1$. Now, let $I_j = I(V_j)$, giving $I_1 \nsupseteq I_2 \nsupseteq I_1$, and $I(V) = I_1 \cap I_2$. Therefore, there exists $f_1 \in I_1 \setminus I_2$ and $f_2 \in I_2 \setminus I_1$. Each f_i is not an element of $I(V)$, but $f_1 f_2 \in I(V)$. So $I(V)$ cannot be prime.

Conversely, suppose $I(V)$ is not prime, so $f_1 f_2 \in I(V)$ but $f_1, f_2 \notin I(V)$. Define $V_1 = V \cap \mathbb{V}(f_1)$ and $V_2 = V \cap \mathbb{V}(f_2)$. Since neither f_i is contained in $I(V)$, $V_i \neq V$. Also, if $P \in V$, we have $f_1(P) f_2(P) = 0$, so $P \in V_1 \cup V_2$. So V is reducible. □

Example. Let $V = \mathbb{V}(XY) \subset \mathbb{A}^2$. Then $V = \mathbb{V}(X) \cup \mathbb{V}(Y)$ is a decomposition of V into irreducible components. Indeed, $\mathbb{V}(X)$ is irreducible, as $I(\mathbb{V}(X)) = (X)$ is a prime ideal in $\mathbb{C}[X, Y]$, and similarly for Y.

2 Structures on varieties

2.1 Coordinate rings

Consider a polynomial $f \in \mathbb{C}[\mathbf{X}]$. We obtain a function $f : \mathbb{A}^n \to \mathbb{A}^1$, If $V \subseteq \mathbb{A}^n$ and $f, g \in \mathbb{C}[\mathbf{X}]$, we are interested in when f, g induce the same set-theoretic function on V. We intend to show that f, g induce the same function if and only if $f - g \in I(V)$. Therefore, we can study polynomials modulo this relation by taking the quotient with respect to this ideal.

Definition. Let $V \subseteq \mathbb{A}^n$ be a variety. The *coordinate ring* of V, or the *ring of regular functions* of V, is defined as ${}^{\mathbb{C}[\mathbf{X}]}\!/\!_{I(V)}$, denoted $\mathbb{C}[V]$ or $\mathcal{O}(V)$.

Corollary. Let V be a variety. Then V is irreducible if and only if $\mathbb{C}[V]$ is an integral domain.

Remark. $\mathbb{C}[V]$ does not precisely determine V or $I(V)$. For instance, consider a surjective homomorphism $\theta : \mathbb{C}[\mathbf{X}] \to \mathbb{C}[V]$, then $\ker\theta = I$ is an ideal, and $\mathbb{V}(I)$ is a variety with coordinate ring $\mathbb{C}[V]$. However, there is not a unique such homomorphism in general. For instance, $\mathbb{C}[X] \simeq {}^{\mathbb{C}[X,Y]}\!/\!_{(Y)}$.

Definition. Let $I \lhd \mathbb{C}[\mathbf{X}]$. We define the *radical ideal* of I to be

$$\sqrt{I} = \{f \in \mathbb{C}[\mathbf{X}] \mid \exists m > 0,\ f^m \in I\}$$

This is an ideal. $\sqrt{\sqrt{I}} = \sqrt{I}$. Note that $\mathbb{V}(I) = \mathbb{V}\left(\sqrt{I}\right)$.

Theorem (strong form of Hilbert's Nullstellensatz)**.** Let $I \lhd \mathbb{C}[\mathbf{X}]$ be an ideal, and $V = \mathbb{V}(I)$. Then $I(V) = \sqrt{I}$.

Therefore, the map $V \mapsto I(V)$ maps precisely onto the space of radical ideals, ideals which are equal to their radicals.

Example. Let $V = \{0\} \in \mathbb{A}^1$. We can write $V = \mathbb{V}(X^2)$, so its coordinate ring is

$$ {}^{\mathbb{C}[X]}\!/\!_{I(\mathbb{V}(X^2))} = {}^{\mathbb{C}[X]}\!/\!_{\sqrt{(X^2)}} = {}^{\mathbb{C}[X]}\!/\!_{(X)} \simeq \mathbb{C}$$

In building the coordinate ring, we forget the structure of X^2. If we had instead considered ${}^{\mathbb{C}[X]}\!/\!_{(X^2)}$, we would have certain nonzero elements whose squares are zero.

2.2 Morphisms

Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be affine varieties.

Definition. A *regular map* or *morphism* from V to W is a function $\varphi : V \to W$ such that there exist elements $f_1, \dots, f_m \in \mathbb{C}[V]$ such that

$$\varphi(P) = (f_1(P), \dots, f_m(P))$$

for all $P \in V$.

The set of all morphisms from V to W is denoted $\mathrm{Mor}(V, W)$.

Example. The morphisms V to $\mathbb{A}^1$ are precisely the functions in the coordinate ring $\mathbb{C}[V]$.

Example. Linear projections $\mathbb{A}^n \to \mathbb{A}^m$ are morphisms. More generally, linear transformations and affine translations are also morphisms.

Example. If $V \subseteq W \subseteq \mathbb{A}^n$ where V, W are varieties, then the inclusion map $V \hookrightarrow W$ is a morphism.

Proposition. Let $\varphi : V \to W, \psi : W \to Z$ be morphisms. Then the composite map $\psi \circ \varphi$ is a morphism $V \to Z$.

Proof. The composition of polynomials is a polynomial. □

2.3 Pullbacks

Definition. Let $\varphi : V \to W$ be a morphism, and let $g \in \mathbb{C}[W]$. Then, the *pullback* is $\varphi^\star(g) = g \circ \varphi : V \to \mathbb{C}$. Note that $\varphi^\star(g) \in \mathbb{C}[V]$, so $\varphi^\star$ gives a map $\mathbb{C}[W] \to \mathbb{C}[V]$.

Remark. This map $\varphi^\star$ is a ring homomorphism, and restricts to the identity on $\mathbb{C}$.

Definition. A ring homomorphism $\mathbb{C}[X] \to \mathbb{C}[Y]$ that restricts to the identity on $\mathbb{C}$ is called a $\mathbb{C}$-*algebra homomorphism*.

Theorem. Let $V \subseteq \mathbb{A}^n, W \subseteq \mathbb{A}^m$ be affine varieties. The map $\alpha : \varphi \mapsto \varphi^\star$ defines a bijection from $\mathrm{Mor}(V, W)$ to the space of $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[W] \to \mathbb{C}[V]$.

Proof. Let $y_1, \dots, y_n \in \mathbb{C}[W]$ be the coordinate functions on W, which are the restrictions of the standard linear coordinate functions on $\mathbb{A}^n$.

First, we show injectivity of α. Let $\varphi : V \to W$ be a morphism. For any point $P \in V$,

$$\varphi(P) = (y_1(\varphi(P)), \dots, y_m(\varphi(P))) = (\varphi^\star(y_1)(P), \dots, \varphi^\star(y_n)(P))$$

So φ is determined by the values of $\varphi^\star(y_1), \dots, \varphi^\star(y_n)$.

Now we show its surjectivity. Let $\lambda : \mathbb{C}[W] \to \mathbb{C}[V]$ be a $\mathbb{C}$-algebra homomorphism, and let $f_i = \lambda(y_i) \in \mathbb{C}[V]$. We can now define the map $\varphi = (f_1, \dots, f_m) : V \to \mathbb{A}^m$. We will show that φ has image contained in W, so that we have $\varphi : V \to W$, which then shows that φ is a morphism $V \to W$. For $P \in V$, we must show $g(\varphi(P)) = 0$ for all $g \in I(W)$. We obtain $g(f_1(P), \dots, f_m(P)) = \lambda(g)(P)$. But $g = 0$ in $\mathbb{C}[W]$, so $g(\varphi(P)) = 0$ as required. Hence $\varphi : V \to W$ is a morphism, and $\lambda = \varphi^\star$ since $\varphi^\star(y_i) = f_i = \lambda(y_i)$. □

Definition. Two affine varieties V, W are *isomorphic* if we have $\varphi : V \to W, \psi : W \to V$ where $\varphi \circ \psi = \mathrm{id}_W$ and $\psi \circ \varphi = \mathrm{id}_V$.

Theorem. V is isomorphic to W if and only if $\mathbb{C}[V]$ is isomorphic to $\mathbb{C}[W]$ as $\mathbb{C}$-algebras.

Proof. Use the previous theorem. □

Example. The affine line $\mathbb{A}^1$ is isomorphic to the twisted cubic $\left\{(t, t^2, t^3) \mid t \in \mathbb{C}\right\}$. This can be easily shown by calculating the coordinate rings explicitly.

Example. Let $V \subseteq \mathbb{A}^2$ be given by $X_1X_2(X_1 - X_2) = 0$. This is the union of three lines, intersecting at the origin. Let $W \subseteq \mathbb{A}^3$ be given by $X_1X_2 = X_2X_3 = X_3X_1 = 0$, which is also a union of three lines, which in this case are the coordinate axes. These are not isomorphic as varieties, because their coordinate rings are not isomorphic, which can be easily shown using tangent spaces, defined in later sections. Note, however, that V and W are homeomorphic in the Euclidean topology.

2.4 Rational functions

Definition. Let $V \subseteq \mathbb{A}^n$ be an irreducible variety. Its *function field*, *field of rational functions*, or *field of meromorphic functions* is the field of fractions $\mathbb{C}(V) = FF(\mathbb{C}[V])$ of $\mathbb{C}[V]$.

Remark. Since V is irreducible, $I(V)$ is prime, so $\mathbb{C}[V]$ is an integral domain. This allows us to construct the field of fractions.

Definition. Let φ be a rational function. A point $P \in V$ is called *regular* if φ can be expressed as a ratio $\frac{f}{g}$ with g$(P) \neq 0$.

Remark. If $\varphi = \frac{f}{g}$, we obtain a well-defined function $\varphi : V \setminus \mathbb{V}(g) \to \mathbb{C}$. The domain is an open set in V, since $\mathbb{V}(g)$ is Zariski closed.

Example. Consider the rational function $X_1^2/X_2 \in \mathbb{C}(\mathbb{A}^2)$. This defines a map on the complement of the X_2-axis. Note that X^3/X_1X_2 defines the same function, but only on points other than $\mathbb{V}(X_1X_2)$. Note that $X^3/X_1X_2 = X_1^2/X_2 \in \mathbb{C}(\mathbb{A}^2)$, so we cannot quite think of elements of $\mathbb{C}(\mathbb{A}^2)$ as functions.

Remark. A rational function on V can be thought of as a pair (U, f) with $U \subseteq V$ Zariski open, such that f is a function $U \to \mathbb{C}$. We define the equivalence relation $(U, f) \sim (U', f')$ if f, f' agree on some nonempty Zariski open set contained in U and U'. Note that if V is irreducible, every nonempty open subset is dense in the Zariski topology.

Definition. A *local ring* is a ring R that contains a unique maximal ideal.

Definition. Let V be an irreducible variety, and let P. The *local ring of V at P* is $\mathcal{O}_{V,P} = \{f \in \mathbb{C}(V) \mid f \text{ is regular at } P\}$.

Proposition. The local ring of an irreducible variety V at a point P is a local ring. Its unique maximal ideal is

$$\mathfrak{m}_{V,P} = \left\{f \in \mathcal{O}_{V,P} \mid f(P) = 0\right\} = \ker(f \mapsto f(P))$$

Further, the invertible elements of $\mathcal{O}_{V,P}$ are precisely those f such that $f(P) \neq 0$.

The proof follows from the following more general lemma.

Lemma. A ring R is a local ring if and only if $R \setminus R^\star$ is an ideal. If so, the unique maximal ideal is $R \setminus R^\star$.

Proof. If $A \trianglelefteq R$ is an ideal, then $A \neq R$ if and only if $A \subseteq R \setminus R^\star$, because if any unit lies in A, it must be all of R. Hence, if $R \setminus R^\star$ is an ideal, it is automatically the unique maximal ideal.

Conversely, let R be a local ring with unique maximal ideal $\mathfrak{m}$. Then $\mathfrak{m} \subseteq R \setminus R^\star$, and if $x \in R \setminus R^\star$ we must have $(x) \neq R$, so $(x) \subseteq \mathfrak{m}$ by maximality. Hence $\mathfrak{m} = R \setminus R^\star$. $\square$

Note that this proves the previous proposition, as $\frac{f}{g} \in \mathcal{O}_{V,P}$ is invertible if and only if $\left(\frac{f}{g}\right)(P) \neq 0$.

Example. Let

$$R = \left\{ \frac{f}{g} \in \mathbb{C}(t) \,\middle|\, \text{ignoring factors, } g(0) \neq 0 \right\} = \mathcal{O}_{\mathbb{A}^1,0}$$

Here, the maximal ideal is (t), and $R/(t) = \mathbb{C}$.

Let $S = \mathbb{C}[\![t]\!]$ be the ring of formal power series in t. This is a local ring by the lemma; its maximal ideal is (t). Note that in fact $R \subseteq S$.

3 Projective varieties

We will construct the projective space $\mathbb{P}^n$, which will be an upgrade to $\mathbb{A}^n$; it is not immediately obvious why $\mathbb{P}^n$ is considered 'better'. Projective space has some interesting properties, such as:

- every pair of lines in $\mathbb{P}^2$ that are distinct meet at a unique point;
- if V is a projective variety (defined shortly) in $\mathbb{P}^2$ defined by a degree d polynomial, if V is a manifold then V is homeomorphic in the Euclidean topology to a closed orientable topological surface of genus $\binom{d-1}{2}$.
- $\mathbb{P}^n$ is compact in the Euclidean topology, but $\mathbb{A}^n$ is not.

3.1 Definition

Definition. Let U be a finite-dimensional complex vector space. The *projectivisation* of U, written $\mathbb{P}(U)$, is the set of lines in U through the origin $\mathbf{0} \in U$. Define $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$.

We usually index the coordinates on $\mathbb{C}^{n+1}$ with indices $0, \dots, n$. A line in $\mathbb{C}^{n+1}$ is therefore given by $\{(a_0 t, \dots, a_n t) \mid t \in \mathbb{C}\}$, and is written $L_{(a_0,\dots,a_n)}$, where not all a_i are zero. We write $(a_0 : a_1 : \dots : a_n)$ for the corresponding element of $\mathbb{P}^n$. Therefore,

$$\mathbb{P}^n = \{(a_0, \dots, a_n) \mid a_i \in \mathbb{C}, \text{not all } a_i = 0\} / \text{scaling by } \mathbb{C}^\star$$

For example, $(2 : 1 : -2) = (4 : 2 : -4) \in \mathbb{P}^2$.

We can decompose $\mathbb{P}^1$ as

$$\begin{aligned} \{(a_0 : a_1) \mid a_0 \neq 0\} \cup \{(a_0 : a_1) \mid a_0 = 0\} &= \{(1 : z) \mid z \in \mathbb{C}\} \cup \{(0 : 1)\} \\ &= \mathbb{A}^1 \cup \text{ a point at infinity} \end{aligned}$$

More generally,

$$\mathbb{P}^n = \{(a_0 : \dots : a_n) \mid a_0 \neq 0\} \cup \{(0 : a_1 : \dots : a_n)\} = \mathbb{A}^n \amalg \mathbb{P}^{n-1}$$

By induction, $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \dots \cup \mathbb{A}^1 \cup$ a point, where the terms other than $\mathbb{A}^n$ are considered 'at infinity'.

Definition. The *Zariski* (respectively *Euclidean*) topology on projective space is the quotient topology for the subspace topology for the Zariski (respectively Euclidean) topology on $\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$, where $\mathbb{P}^n = {\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}}/{\sim}$ and $\mathbb{C}^{n+1} \setminus \{0\} \subseteq \mathbb{C}^{n+1}$.

There is a copy of S^{2n+1} inside $\mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$, which therefore surjects onto $\mathbb{P}^n$.

Corollary. $\mathbb{P}^n$ is compact.

Proof. It is the continuous image of the compact set S^{2n+1}. □

Definition. For $0 \leq j \leq n$, we define the *jth coordinate hyperplane* is the set $H_j = \{(\mathbf{a}_i) \mid a_j = 0\} \subseteq \mathbb{P}^n$.

We can naturally identify H_j with $\mathbb{P}^{n-1}$.

Definition. The *jth standard affine patch* U_j is the complement of H_j.

There is a natural bijection $U_j \to \mathbb{A}^n$ by mapping $(a_0 : \dots : a_n)$ to $\left(\frac{a_0}{a_j}, \dots, \widehat{\frac{a_j}{a_j}}, \dots, \frac{a_n}{a_j}\right)$ where the hat denotes 'forgetting' that element of the tuple. The inverse function maps $(b_1, \dots, b_n)$ to $(b_1 : \dots : b_{j-1} : 1 : b_j : \dots : b_n)$. We observe that $\{U_j\}_{j=0}^n$ is an open cover of $\mathbb{P}^n$ in the Zariski topology.

3.2 Projective varieties

Example. Consider the polynomial $X_0 + 1 \in \mathbb{C}[X_0, X_1]$. Note that $X_0 + 1$ does not define a function on $\mathbb{P}^1$. For example, $(-1 : 0) = (1 : 0)$, but $X_0 + 1$ vanishes on the first representative and not the second. The vanishing locus of $X_0 + 1$ on $\mathbb{P}^1$ is therefore undefined. Therefore, we need a slightly more subtle definition of a variety in projective space.

Definition. A *monomial* in $\mathbb{C}[\mathbf{X}] = \mathbb{C}[X_0, \dots, X_n]$ is an element of the form $X_0^{d_0} X_1^{d_1} \dots X_n^{d_n}$ where $d_i \geq 0$. A *term* is a nonzero multiple of a monomial. The *degree* of a term $cX_0^{d_0} \dots X_n^{d_n}$ is $\sum_{i=0}^n d_i$. A *homogeneous polynomial* of degree d is a finite sum of terms of degree d.

Any polynomial can be uniquely decomposed as a sum of homogeneous polynomials of different degree; we write $f = \sum_{i=0}^\infty f_{[i]}$ where the $f_{[i]}$ are homogeneous of degree i. Note that this sum is always finite.

Lemma. Let $f \in \mathbb{C}[\mathbf{X}]$ be homogeneous, and let $(a_0, \dots, a_n) \in \mathbb{C}^{n+1} \setminus \{\mathbf{0}\}$. Then, if $f(\mathbf{a}) = 0$, we have $f(\lambda \mathbf{a}) = 0$ for all $\lambda \in \mathbb{C}^\star$.

Proof. Trivial by checking the definitions. □

Corollary. Let $f \in \mathbb{C}[\mathbf{X}]$ be homogeneous. Then

$$\mathbb{V}(f) = \{P \in \mathbb{P}^n \mid f(\mathbf{a}) = 0 \text{ for any (or every) representative of } P\}$$

is well-defined.

Definition. An ideal $I \trianglelefteq \mathbb{C}[\mathbf{X}]$ is called *homogeneous* if it can be generated by homogeneous polynomials (of potentially different degrees).

Lemma. Let $I \trianglelefteq \mathbb{C}[\mathbf{X}]$ be an ideal. Then I is homogeneous if and only if whenever $f \in I$, all of the homogeneous parts $f_{[r]}$ are also contained in I.

Proof. Suppose I is homogeneous. Then let g_j be homogeneous generators of I of degree d_j. Writing $f = \sum h_j g_j$ for arbitrary $h_j \in \mathbb{C}[\mathbf{X}]$, we can split each h_j into its pieces $h_{j[r]}$. Now, $h_{j[r]} g_j \in I$ is homogeneous, and its degree is rd_j. Hence, $f_{[r]} = \sum_j h_{j[r-dj]} g_j \in I$ as required. The converse is trivial by decomposing the generators of the ideal. □

Definition. Let $I \trianglelefteq \mathbb{C}[\mathbf{X}]$ be a homogeneous ideal. Then, the *vanishing locus* is $\mathbb{V}(I) = \{P = (\mathbf{a}_i) \in \mathbb{P}^n \mid \forall f \in I,\ f((\mathbf{a}_i)) = 0\}$. A *projective variety* in $\mathbb{P}^n$ is any set of this form.

Note that we could have defined the vanishing locus using the quantifier 'for all *homogeneous* $f \in I$'.

Example. Let $U \subseteq \mathbb{C}^{n+1}$ be any vector subspace. Let the projectivisation of U is a subset of $\mathbb{P}^n$, and is a projective variety. More concretely, $U = \left\{\mathbf{v} \in \mathbb{C}^{n+1} \mid \forall j,\ \sum_{i=0}^n a_i^{(j)} v_i = 0\right\}$ for a subset $\mathbf{a}^{(j)} = (a_0^{(j)}, \dots, a_n^{(j)})$, as a vector subspace is the kernel of some linear map. Therefore, $\mathbb{P}(U) = \mathbb{V}(I)$ where I is the ideal generated by $F_j = \sum_i a_i^{(j)} X_i \in \mathbb{C}[\mathbf{X}]$. More generally, a projective linear space is the projectivisation of a linear subspace. Hence, projective linear spaces in $\mathbb{P}^n$ are in bijection with linear subspaces in $\mathbb{C}^{n+1}$.

$GL_{n+1}(\mathbb{C})$ acts on $\mathbb{P}^n$ coordinatewise. The normal subgroup of scalar matrices $\mathbb{C}^\star \subseteq GL_{n+1}(\mathbb{C})$ acts trivially on $\mathbb{P}^n$. The quotient is written $PGL_n(\mathbb{C}) = {}^{GL_{n+1}(\mathbb{C})}\!/\!_{\mathbb{C}^\star}$, and acts transitively on $\mathbb{P}^n$.

Example. The *Segre surface* is the hypersurface $S_{11} = \mathbb{V}(X_0X_3 - X_1X_2) \subseteq \mathbb{P}^3$. Consider the map $\sigma_{11} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ given by $\sigma_{11}((a_0 : a_1), (b_0 : b_1)) = (a_0b_0 : a_0b_1 : a_1b_0 : a_1b_1)$. One can show that this map is well-defined, and in fact, $\operatorname{Im} \sigma_{11} = S_{11}$.

First, consider the map $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^4$ where we identify $\mathbb{C}^4$ with the space of 2×2 matrices on $\mathbb{C}$, given by the outer product. More precisely, $(v, w) \mapsto vw^\intercal$. The image of this map is precisely the set of matrices of rank at most 1. Hence, the image is the vanishing locus of $X_0X_3 - X_1X_2$, the determinant of such a matrix.

3.3 Homogenisation and projective closure

Recall that $\mathbb{P}^n = U_0 \cup \dots \cup U_n$, where $U_i = \mathbb{P}^n \setminus \mathbb{V}(X_i)$. We therefore have the following different descriptions of a Zariski topology on $\mathbb{P}^n$:

(i) the quotient of the subspace of the Zariski topology on $\mathbb{C}^{n+1}$;

(ii) define that V is Zariski-closed if and only if $V = \mathbb{V}(I)$ where $I \lhd \mathbb{C}[\mathbf{X}]$ is homogeneous;

(iii) the gluing topology: define that a subset $Z \subseteq \mathbb{P}^n$ is closed if $Z \cap U_i$ is closed for all i, as the U_i are isomorphic to $\mathbb{A}^n$.

These three constructions coincide.

If $V \subseteq \mathbb{P}^n$ is a projective variety, consider $U_0 \cap V \subseteq U_0$. If $V = \mathbb{V}(I)$, then $U_0 \cap V = \mathbb{V}(I_0)$ where $I_0 = \{f = F(1, Y_1, \dots, Y_n) \mid F \in I \text{ homogeneous}\} \subseteq \mathbb{C}[Y_1, \dots, Y_n]$. Identifying U_0 with $\mathbb{A}^n$ with coordinates $Y_1, \dots, Y_n$ (so $Y_j = \frac{X_j}{X_0}$), $V \cap U_0$ is naturally an affine variety.

Conversely, let $W \subseteq \mathbb{A}^n$ be an affine variety, and identify $\mathbb{A}^n$ with U_0. Then, the Zariski closure $\overline{W}$ of W inside $\mathbb{P}^n$ is a projective variety. We are interested in studying the precise projective varieties obtained in this way.

Definition. Let $f \in \mathbb{C}[Y_1, \dots, Y_n]$ be an arbitrary polynomial of total degree d. The *homogenisation* of f, written F or f^h, is

$$f^h(X_0, \dots, X_n) = X_0^d f\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right) \in \mathbb{C}[X_0, \dots, X_n]$$

This is homogeneous of degree d. Similarly, if I is an ideal in $\mathbb{C}[Y_1, \dots, Y_n]$, its homogenisation $I^\star = I^h$ is the ideal generated by the homogenisation of the elements $f \in I$; this is a homogeneous ideal in $\mathbb{C}[X_0, \dots, X_n]$. Given an affine variety $V \subseteq \mathbb{A}^n$, the *projective closure* of V is $\mathbb{V}(I(V)^h) \subseteq \mathbb{P}^n$.

Example. Let $f(Y_1, Y_2) = 1 + Y_1^2 + Y_1 Y_2^2$. Its homogenisation is $f^h(X_0, X_1, X_2) = X_0^3 + X_0 X_1^2 + X_1 X_2^2$.

Remark. Let $I = (f_1, \dots, f_r) \subseteq \mathbb{C}[Y_1, \dots, Y_n]$, and let $J = (f_1^h, \dots, f_r^h)$. Typically, $J \neq I^h$. If I is principal, this holds: $I = (f)$ implies $I^h = (f^h)$.

Proposition. Let $V \subseteq \mathbb{A}^n$ be an affine variety. Then, the Zariski closure $\overline{V} \subseteq \mathbb{P}^n$ given by identifying $U_0 = \mathbb{A}^n$ coincides with the projective closure $\mathbb{V}(I(V)^h) \subseteq \mathbb{P}^n$.

Proof. Let I be an ideal in $\mathbb{C}[Y_1, \dots, Y_n]$, and let $V = \mathbb{V}(I)$. Let $\overline{V}$ be the Zariski closure. Let I^h be the homogenisation of the ideal. Then $\mathbb{V}(I^h)$ is Zariski closed, and contains V. We will show that this is the smallest such set.

Suppose $Y \supseteq V$ is closed, so $Y = \mathbb{V}(I')$ where I' is homogeneous. Any homogeneous element in I' can be written as $X_0^d f^h$ for some $f \in \mathbb{C}[Y_1, \dots, Y_n]$. Now, $X_0^d f^h = 0$ on $V \subseteq \mathbb{P}^n$, so $f = 0$ on $V \subseteq \mathbb{A}^n$. Hence $f \in I(V) = \sqrt{I}$ by the Nullstellensatz. So $f^m \in I$ for some $m > 0$, so $(f^m)^h = (f^h)^m \in I^h$. Hence $f^h \in \sqrt{I^h}$, so $X_0^d f^h \in \sqrt{I^h}$. Therefore, $I' \subseteq \sqrt{I^h}$. $\square$

Remark. Let $V \subseteq \mathbb{P}^n$, and let $W = V \cap U_0 \subseteq \mathbb{A}^n$. Then $\overline{W} \subseteq \mathbb{P}^n$ is not in general equal to V. For example, let $V = \mathbb{V}(X_0)$, so $W = \varnothing$ and $\overline{W} = \varnothing$. This ambiguity arises due to the X_0^d term required in the above proof when dehomogenising a polynomial.

This shows that the topological notion of the Zariski closure and the algebraic notion of the projective closure agree.

Example. Let $V \subseteq \mathbb{P}^2$ be given by $\mathbb{V}(X_0X_1 - X_2^2)$. We obtain $V_0 \subseteq U_0$ given by setting $X_0 = 1$, $V_1 \subseteq U_1$ given by setting $X_1 = 1$, and $V_2 \subseteq U_2$ given by setting $X_2 = 1$. We find $V_0 = \mathbb{V}(Y_1 - Y_2^2)$ which is a parabola, and V_1 is similar. $V_2 = \mathbb{V}(X_0X_1 - 1)$, which is a rectangular hyperbola.

Theorem. Let $Q \subseteq \mathbb{P}^n$ be given by $\mathbb{V}(f)$ where f is a homogeneous quadratic polynomial. Then, after a change of coordinates $A \in PGL_n(\mathbb{C})$, Q has the form $\mathbb{V}(X_0^2 + \dots + X_r^2)$ where r is the rank of the quadratic form f.

Proof. Use the results from IB Linear Algebra. □

Theorem (projective Nullstellensatz)**.** If $\mathbb{V}(I) = \varnothing \subseteq \mathbb{P}^n$ where I is a homogeneous ideal, then $I \supseteq (X_0^m, \dots, X_n^m)$ for some $m \in \mathbb{N}$. Further, if $V = \mathbb{V}(I) \neq \varnothing$, then $I^h(V) = \sqrt{I}$, where $I^h(V)$ is the ideal generated by homogeneous polynomials vanishing on V.

Proof. We reduce to the affine case. Let I be a homogeneous ideal, and let $V^a = \mathbb{V}(I) \subseteq \mathbb{A}^{n+1}$. Note that $\mathbf{0} \in V^a$, assuming $V \neq \varnothing$. Then there is a continuous map $V^a \setminus \{0\} \to V$ obtained by the restriction of $\mathbb{A}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{P}^n$. Moreover, this map is surjective, so is a quotient map. Note that V is empty if and only if $V^a = \{\mathbf{0}\}$. So the result holds by the affine Nullstellensatz. The second part is similar. □

Let V be a projective variety in $\mathbb{P}^n$. If $W \subseteq V$ is a variety closed in V, we say W is a *closed subvariety* of V. The complement $V \setminus W$ is an *open subvariety*. The closed (respectively open) subvarieties of V satisfy the axioms of the closed (open) sets of a topology. We say V is irreducible if V cannot be written as $V_1 \cup V_2$ for proper closed subvarieties V_1, V_2.

Proposition. (i) Every projective variety is a finite union of irreducible varieties.
(ii) V is irreducible if and only if $I^h(V)$ is prime.

Proof. Part (i) is identical to the affine case. For part (ii), first observe that if I is a homogeneous ideal which is not prime, we can find homogeneous $F, G \notin I$ such that $FG \in I$, as I is generated by homogeneous elements. Then the proof for the affine case works as before. □

Let $S \subseteq V$ be a subset. S is Zariski dense in V if and only if every homogeneous polynomial that vanishes on S vanishes on V.

Proposition. Let $V \subseteq \mathbb{P}^n$ be an irreducible projective variety. Let $W \subsetneq V$ be a proper closed subvariety. Then, $V \setminus W$ is dense in V.

Intuitively, W is lower-dimensional than V, and V with a lower-dimensional set removed is dense.

Proof. Let $f \in \mathbb{C}[\mathbf{X}]$ be a homogeneous polynomial that vanishes on $V \setminus W$. As $W \neq V$, there exists a polynomial $g \in I^h(W) \setminus I^h(V)$ by the projective Nullstellensatz. Then, fg vanishes on all of V. But $I^h(V)$ is prime as V is irreducible, so $f \in I^h(V)$. □

3.4 Rational functions

Homogeneous polynomials have well-defined zero sets in $\mathbb{P}^n$, but not a well-defined value. Therefore, we cannot define a coordinate ring $\mathbb{C}[V]$ in an analogous way. However, a ratio of homogeneous polynomials of the same degree does have a well-defined value on $\mathbb{P}^n$ away from the vanishing locus of the denominator.

Definition. Let $V \subseteq \mathbb{P}^n$ be an irreducible projective variety. The *function field* or *field of rational functions* is

$$\mathbb{C}(V) = \left\{ \frac{F}{G} \,\middle|\, F, G \in \mathbb{C}[\mathbf{X}] \text{ homogeneous and have the same degree, } G \notin I^h(V) \right\} \Big/ \sim$$

where $\frac{F_1}{G_1} \sim \frac{F_2}{G_2}$ if $F_1 G_2 - F_2 G_1 \in I^h(V)$.

Lemma. The relation $\sim$ is an equivalence relation.

Proof. Reflexivity and symmetry are clear. Now suppose that $\frac{F_1}{G_1} \sim \frac{F_2}{G_2} \sim \frac{F_3}{G_3}$, so $F_2G_1 - F_1G_2 \in I^h(V)$ and $F_3G_2 - F_2G_3 \in I^h(V)$. Consider $F_1G_3 - F_3G_1$. Multiplying by G_2, $F_1G_2G_3 - F_3G_1G_2$. Since $G_2 \notin I^h(V)$, primality of $I^h(V)$ implies that it suffices to show $F_1G_2G_3 - F_3G_1G_2 \in I^h(V)$. In the ring $\mathbb{C}[\mathbf{X}] / I^h(V)$, we have relations $F_1G_2 = F_2G_1$ and $F_3G_2 = F_2G_3$. Hence $F_1G_2G_3 - F_3G_1G_2 = 0$ in $\mathbb{C}[\mathbf{X}] / I^h(V)$. □

Note that $\mathbb{C}(V)$ is a field.

Proposition. The field $\mathbb{C}(V)$ is a finitely generated field extension of $\mathbb{C}$.

Note that $\mathbb{C}(t)$ is finitely generated as a field, but not finitely generated as a $\mathbb{C}$-module or a $\mathbb{C}$-algebra.

Proof. Suppose $V \neq \varnothing$. Then, there is some coordinate function X_i that is nonzero on V; without loss of generality let $i = 0$. We claim that $\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}$ generate $\mathbb{C}(V)$ over $\mathbb{C}$. Explicitly, if $\frac{F}{G}$ is a degree 0 ratio, it can be written in terms of the $\frac{X_j}{X_0}$ and the field operations. It suffices to show the result holds when $\frac{F}{G}$ is of the form $\frac{M}{G}$ where M is a monomial. Then, it suffices to show the result for $\frac{G}{M}$ where M is a monomial by taking reciprocals. Hence, it suffices to show the result for $\frac{M}{M'}$ where M, M' are monomials, and this is trivial. □

Corollary. Let $V \subseteq \mathbb{P}^n$ be an irreducible projective variety, not contained in the hyperplane $\{X_0 = 0\}$. Let $V_0 = V \cap U_0$, where $U_0 \simeq \mathbb{A}^n$ is the first affine patch. Then, $\mathbb{C}(V_0) = \mathbb{C}(V)$, where $\mathbb{C}(V_0) = FF(\mathbb{C}[V_0])$.

Proof. V_0 has coordinate ring

$$\mathbb{C}\left[\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right]\Big/ I(V_0)$$

Hence, $\mathbb{C}(V_0) = FF(\mathbb{C}[V_0])$ is generated by the $\frac{X_j}{X_0}$. □

Definition. Let $\varphi \in \mathbb{C}(V)$ and let $P \in V$. We say that φ is *regular* or *defined* at P if φ can be expressed as $\frac{F}{G}$ where F, G are homogeneous of the same degree with $G(P) \neq 0$. There is a partial function from the set of regular points of V to $\mathbb{C}$.

Definition. The *local ring* of V at P, written $\mathcal{O}_{V,P}$, is the set of $\varphi \in \mathbb{C}(V)$ such that φ is regular at P. This is a subring of $\mathbb{C}(V)$, which is a local ring in the sense of commutative algebra.

Proposition. Let $V \subseteq \mathbb{P}^n$ be an irreducible projective variety not contained in $\{X_0 = 0\}$. Let $V_0 = V \cap U_0$ where $U_0 = \{X_0 = 0\}$. Let P be a point in V_0. Then, there is a natural isomorphism $\mathcal{O}_{V,P} \to \mathcal{O}_{V_0,P}$ respecting the isomorphism $\mathbb{C}(V) \simeq \mathbb{C}(V_0)$.

Proof. Follows by unfolding the definitions. □

3.5 Rational maps

Let $F_0, \dots, F_m \in \mathbb{C}[\mathbf{X}] = \mathbb{C}[X_0, \dots, X_n]$ be homogeneous of the same degree d. Define $\mathbf{F} = (F_0, \dots, F_m) \colon \mathbb{C}^{n+1} \to \mathbb{C}^{m+1}$.

Proposition. The map $\mathbf{F}$ descends to a well-defined map of sets $\varphi \colon \mathbb{P}^n \setminus \bigcap_j \mathbb{V}(F_j) \to \mathbb{P}^m$. If P is represented by $\mathbf{a} = (a_0, \dots, a_n)$, then $\varphi(P)$ is represented by $(F_0(\mathbf{a}), \dots, F_m(\mathbf{a}))$.

Proof. Since all F_j are homogeneous of the same degree d, $\lambda\mathbf{a} = (\lambda a_0, \dots, \lambda a_n)$ gives

$$(F_0(\lambda\mathbf{a}), \dots, F_m(\lambda\mathbf{a})) = \lambda^d(F_0(\mathbf{a}), \dots, F_m(\mathbf{a}))$$

which is equivalent to $\varphi(P)$. □

We will denote such maps $\mathbf{F} = (F_0, \dots, F_m)$ by $\varphi \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^m$.

Let G be a nonzero homogeneous polynomial in $X_0, \dots, X_n$. Given $\mathbf{F} \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, we can also consider $G\mathbf{F} = (GF_0, \dots, GF_n) \colon \mathbb{P}^n \dashrightarrow \mathbb{P}^m$. Observe that the maps $\mathbf{F}$ and $G\mathbf{F}$ have different domains, but coincide at points where they are both defined. Note that there is a 'best' representative $\mathbf{F}$, as $\mathbb{C}[\mathbf{X}]$

is a unique factorisation domain, but we will not use this notion here, because not all rings that we will use are unique factorisation domains.

Definition. Let $V \subseteq \mathbb{P}^n$ be an irreducible projective variety. Let $F_0, \dots, F_m$ be homogeneous polynomials in $\mathbb{C}[X_0, \dots, X_n]$ of fixed degree d, and not all contained in $I^h(V)$. They determine a map of sets $V \setminus \bigcap_j \mathbb{V}(F_j) \to \mathbb{P}^n$ by the previous construction. Two such tuples $(F_0, \dots, F_m)$ and $(G_0, \dots, G_m)$ are said to *determine the same map* if $F_i G_j - F_j G_i \in I^h(V)$. A *rational map* from V to $\mathbb{P}^m$ is an equivalence class of tuples $(F_0, \dots, F_m)$ as above, where two tuples are equivalent if they determine the same map.

Definition. A point $P \in V$ is a *regular point* of a rational map $\varphi : V \dashrightarrow \mathbb{P}^n$ if there is a representative $(F_0, \dots, F_m)$ of φ such that $F_j(P) \neq 0$ for some j. The *domain* of φ is the set of regular points of φ. A rational map φ is called a *morphism* if the domain of φ is V; in this case, we write $V \to \mathbb{P}^m$.

Example. A linear map $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ is given by an $(m+1) \times (n+1)$ matrix $A = (a_{ij})$. Concretely, we can define $\varphi = (F_0, \dots, F_m)$ where $F_j = \sum_i a_{ij} X_i$. If A has rank $n+1 \leq m+1$, then φ is a morphism.

Example (projection from a point). Let $P = (0 : 0 : 1) \in \mathbb{P}^2$. The *projection from P* is the rational map $\pi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ defined by $(a_0 : a_1 : a_2) \mapsto (a_0 : a_1)$. π is not regular at P, and regular everywhere else.

Let $C = \mathbb{V}(f_d)$ where f_d is a homogeneous polynomial of degree d. Suppose that $P \notin C$. The composition is therefore a morphism $\varpi : C \to \mathbb{P}^1$. One can show that for almost all choices of $Q \in \mathbb{P}^1$, the fibre $\varpi^{-1}(Q)$ is a set of size d.

Example. Let $C = \mathbb{V}(X_0 X_2 - X_1^2) \subseteq \mathbb{P}^2$. Consider the projection from $(0 : 0 : 1)$, and restrict this projection to C to obtain a map $\pi : C \dashrightarrow \mathbb{P}^1$ defined by $\pi(a_0 : a_1 : a_2) = (a_0 : a_1)$. By changing representatives, we can show π is a morphism, even though $(0 : 0 : 1) \in C$.

The map π is determined by (X_0, X_1); we must look for other pairs (F_0, F_1) that determine the same rational map as π, so $F_0 X_1 - F_1 X_0 \in I^h(C) = (X_0 X_2 - X_1^2)$. Notice that this relation is satisfied by (X_1, X_2), so π agrees with the function $\pi'(a_0 : a_1 : a_2) = (a_1 : a_2)$ on C. So π is regular at $(0 : 0 : 1) \in C$, so π is a morphism.

Observe that for $\pi : C \to \mathbb{P}^1$, $\pi^{-1}(q)$ is a single point for $q \in \mathbb{P}^1$. One can show also that π is surjective.

If W is a projective variety, a rational map (or morphism) $V \to W$ is a rational map (or morphism) $V \to \mathbb{P}^m$ with image contained in W. A morphism $\varphi : V \to W$ is an isomorphism if it has a two-sided inverse morphism.

Proposition. Let C be the vanishing locus of a homogeneous polynomial $f \in \mathbb{C}[X_0, X_1, X_2]$ of degree 2 in $\mathbb{P}^2$. Then, if f is irreducible then $C \simeq \mathbb{P}^1$.

Proof. By changing coordinates, we can assume $f = X_0 X_2 - X_1^2$; the rank of the quadratic form is 2 as f is irreducible. By the example above, we have a morphism $\pi : C \to \mathbb{P}^1$ by projection from $(0 : 0 : 1)$. We define an inverse map $\mu : \mathbb{P}^1 \to \mathbb{P}^2$ by $\mu(Y_0 : Y_1) = (Y_0^2 : Y_0 Y_1 : Y_1^2)$. The image of μ lies in C, and the compositions are inverse. $\square$

There is only one conic in two-dimensional projective space, up to changing coordinates.

Example (Cremona transformation)**.** Consider the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by

$$\kappa(X_0 : X_1 : X_2) = (X_1X_2 : X_0X_2 : X_0X_1)$$

This can be thought of as a coordinatewise reciprocal map. The Cremona transformation maps lines into conics. Suppose ℓ is a line not given by the vanishing locus of any of the coordinate functions X_i. Then, consider the subset $\kappa(\operatorname{dom}\kappa \cap \ell) \subseteq \mathbb{P}^2$; this is the analogue of the image in the case of rational maps. One can show that the closure of this set is a conic.

Example (Veronese embedding)**.** Let $F_0, \dots, F_m$ be the list of monomials of degree d in $X_0, \dots, X_n$, so $m = \binom{n+d}{d} - 1$. We define the $\nu_d : \mathbb{P}^n \to \mathbb{P}^m$ mapping $(\mathbf{a})$ to $(F_0(\mathbf{a}), \dots, F_m(\mathbf{a}))$. One can show this is a morphism. Note that the map $\mu(Y_0 : Y_1) = (Y_0^2 : Y_0Y_1 : Y_1^2)$ used in the previous proposition is an instance of this embedding. In general, ν_d is injective, and the image of ν_d is a projective variety isomorphic to $\mathbb{P}^n$. This fact has a straightforward but tedious proof.

Note that $\mathbb{P}^n \times \mathbb{P}^m \not\simeq \mathbb{P}^{n+m}$.

Example (Segre embedding)**.** Let $n, m > 0$ be integers. The *Segre embedding* is the map $\sigma_{mn} : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{mn+m+n}$ defined by $\sigma_{mn}((x_i), (y_j)) = (x_iy_j)$. We label the coordinates of $\mathbb{P}^{mn+m+n}$ using Z_{ij} for $0 \leq i \leq m$ and $0 \leq j \leq n$. Note that $(m+1)(n+1) - 1$; we have a map $U \times V \to U \otimes V$ and then take the projectivisation, giving the correct dimension.

Theorem. The map σ_{mn} is a bijection between $\mathbb{P}^m \times \mathbb{P}^n$ and the projective variety $\mathbb{V}(I)$ where I is the ideal generated by the $Z_{ij}Z_{pq} - Z_{iq}Z_{pj}$.

Proof. Clearly, $\sigma_{mn}(\mathbb{P}^m \times \mathbb{P}^n) \subseteq V = \mathbb{V}(I)$. Now, consider the affine piece $V_{00} = V \cap \{Z_{00} \neq 0\} \subseteq \mathbb{A}^{mn+m+n}$. The inhomogeneous ideal defining V_{00} is generated by $Y_{ij} - Y_{i0}Y_{0j}$ where $1 \leq i \leq m$ and $1 \leq j \leq n$, and $Y_{ij} = \frac{Z_{ij}}{Z_{00}}$. Note that elements $Y_{ij}Y_{pq} - Y_{iq}Y_{pj}$ for other indices automatically lie in this ideal. On this patch, σ_{mn} defines a morphism $\mathbb{A}^m \times \mathbb{A}^n \to \mathbb{V}(I_{00})$. There is an inverse $\mathbb{A}^{mn+m+n} \to \mathbb{A}^m \times \mathbb{A}^n$, given by

$$(Y_{ij}) \mapsto ((Y_{10}, \dots, Y_{m0}), (Y_{01}, \dots, Y_{0n}))$$

One can check that this is indeed an inverse; this process can be repeated for all other patches $\{Z_{ij} \neq 0\}$, so σ_{mn} is as claimed. $\square$

Hence, if V, W are projective varieties, $V \times W$ is naturally also a projective variety.

3.6 Composition of rational maps

Let $\varphi : V \dashrightarrow W$ and $\psi : W \dashrightarrow Z$ be rational maps between irreducible varieties. The composition $\psi \circ \varphi$ of rational maps may not be well-defined, as the image of the domain of φ could lie entirely inside the locus of indeterminacy of ψ.

Definition. A rational map $\varphi : V \dashrightarrow W$ is *dominant* if $\varphi(\operatorname{dom}\varphi)$ is Zariski dense in W.

Proposition. If φ is dominant, then $\psi \circ \varphi$ is well-defined for any rational map $\psi : W \dashrightarrow Z$.

Proof. Let U denote a dense open set in $\operatorname{dom}\varphi$, and let U' be a dense open set in $\operatorname{dom}\psi$. Then, let $U'' = U \cap \varphi^{-1}(U')$, which is open in V. The composition $\psi\circ\varphi$ is well-defined on U''. This is a rational map as the composition of polynomials is a polynomial. ☐

Definition. If $\varphi : V \dashrightarrow W$ and $\psi : W \dashrightarrow V$ are such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are equivalent to the identity map on W and V respectively, we say that V and W are *birational* and that φ and ψ are *birational maps*.

Example. Any isomorphism is birational.

Example. Consider the Cremona map $\kappa : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined as above by $(x_0 : x_1 : x_2) \mapsto (x_1x_2 : x_0x_2 : x_0x_1)$. Intuitively, $(x_0 : x_1 : x_2) \mapsto \left(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right)$. Then κ is self-inverse as a rational map, hence birational. It is not an isomorphism as it is not defined everywhere.

Remark. One can construct the group $\mathrm{Bir}(\mathbb{P}^2)$ of birational automorphisms of $\mathbb{P}^2$. This group contains a copy of $PGL_2(\mathbb{C})$ and the subgroup generated by κ above.

Theorem. Let V, W be irreducible projective varieties. Then V is birational to W if and only if $\mathbb{C}(V)$ and $\mathbb{C}(W)$ are isomorphic as fields.

Recall the similar result that if V, W are affine varieties, V is isomorphic to W if and only if $\mathbb{C}[V]$ and $\mathbb{C}[W]$ are isomorphic as $\mathbb{C}$-algebras.

Proof. Suppose first that V is birational to W, so $\varphi : V \dashrightarrow W$ is a birational map. Let $f \in \mathbb{C}(W)$. Then, f gives a function $W \dashrightarrow \mathbb{A}^1 = \mathbb{C}$, and composition gives a map of fields $\varphi^\star : \mathbb{C}(W) \to \mathbb{C}(V)$ defined by $f \mapsto f \circ \varphi$. Similarly, φ^{-1} gives a map $\mathbb{C}(V) \to \mathbb{C}(W)$, and the compositions are identical, so we obtain an isomorphism of fields.

For the converse, suppose we have $\mathbb{C}(V) \simeq \mathbb{C}(W)$ as fields. Suppose that $V \subseteq \mathbb{P}^n$ is not contained in $\{X_0 = 0\}$, and $W \subseteq \mathbb{P}^m$ is not contained in $\{Y_0 = 0\}$. We have shown that $\mathbb{C}(V) = \mathbb{C}(x_1, \dots, x_n)$ where x_i is the rational function determined by $\frac{X_i}{X_0}$. Similarly, $\mathbb{C}(W) = \mathbb{C}(y_1, \dots, y_m)$ where y_j is determined by $\frac{Y_j}{Y_0}$.

An isomorphism $\mathbb{C}(V) \simeq \mathbb{C}(W)$ identifies each y_j with $f_j(\mathbf{x})$ where f_j is a rational function in n variables. Writing each $f_j(\mathbf{x})$ as a rational function in the $\frac{X_i}{X_0}$, we can clear denominators by multiplying by some polynomial in the $\frac{X_i}{X_0}$ and homogenise with respect to X_0. We then obtain homogeneous polynomials $F_0, \dots, F_m$ in $X_0, \dots, X_n$ such that

$$f_j\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right) = \frac{F_j(X_0, \dots, X_n)}{F_0(X_0, \dots, X_n)}$$

Now, $F_0, \dots, F_m$ determine a rational map $V \dashrightarrow W$. This can be repeated with the x_i and y_j reversed to obtain a rational map $W \dashrightarrow V$. One can show that these are inverses. ☐

4 Dimension

4.1 Tangent spaces

Let $V \subseteq \mathbb{A}^n$ be an affine hypersurface, so $V = \mathbb{V}(f)$. We assume that f is irreducible, so V is also irreducible. Let $P = (a_1, \dots, a_n) = (\mathbf{a}) \in V$. An affine line through P has the form $L = \{(a_1 + b_1 t, \dots, a_n + b_n t) \mid t \in \mathbb{C}\}$ for $(\mathbf{b}) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ is fixed.

The intersection $V \cap L$ is the set of points on L where f vanishes. This gives $0 = f(a_1 + b_1 t, \dots, a_n + b_n t) = g(t) = \sum_r c_r t^r$, a polynomial in t. Since $P \in V \cap L$, $c_0 = 0$. The linear term c_1 is given by $c_1 = \sum_i b_i \frac{\partial f}{\partial X_i}$. Geometric tangency of L to V is equivalent to the statement that $c_1 = 0$.

Definition. The line L through P is *tangent* to $V = \mathbb{V}(f)$ at P if it is contained in the *tangent space* of V at P, defined by $T^{\text{aff}}_{V,P} = \mathbb{V}(g) \subseteq \mathbb{A}^n$ where

$$g = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial X_i}(P) \right)(X_i - a_i)$$

Note that g is linear. $T^{\text{aff}}_{V,P}$ is n-dimensional if $g = 0$ and $(n-1)$-dimensional otherwise, taking the dimensions as an affine space.

Definition. If $\dim T^{\text{aff}}_{V,P} = n$, we say that P is a *singular point* of V. Otherwise, it is a *smooth point*.

Example (nodal cubic)**.** Consider the affine hypersurface $C = \mathbb{V}(Y^2 - X^2(X+1))$. One can show by direct calculation that the only singular point is $(0,0)$.

Example (cusp)**.** Consider $C = \mathbb{V}(Y^2 - X^3)$. Again, the point $(0,0)$ is the only singular point.

Let $V \subseteq \mathbb{V}(F) \subseteq \mathbb{P}^n$ for F an irreducible homogeneous polynomial.

Definition. The *projective tangent space* to V at P is $T^{\text{proj}}_{V,P} = \mathbb{V}(G)$ where

$$G = \sum_{i=0}^{n} \left(\frac{\partial F}{\partial X_i}(P) \right) X_i$$

To see that $P \in \mathbb{V}(G)$, note that $F(X_0, \dots, X_n) = \frac{1}{\deg F} \sum_{i=0}^n X_i \frac{\partial F}{\partial X_i}$; this is sometimes known as *Euler's formula*. Smooth points and singular points are defined as in the affine case. From the inverse function theorem, if all points are smooth, the tangent space is a manifold.

The affine and projective tangent spaces are compatible in a particular sense. Let $V = \mathbb{V}(F) \not\subseteq \{X_0 = 0\}$, and consider $V_0 = V \cap U_0$. If $P \in V_0 \subseteq V$, we can compute $T^{\text{proj}}_{V,P} \cap U_0$ and $T^{\text{aff}}_{V_0,P}$, which are both subsets of $\mathbb{A}^n$. Let $V_0 = \mathbb{V}(f)$, then $F(X_0, \dots, X_n) = X_0^{\deg F} f\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right)$. By computing $\frac{\partial F}{\partial X_i}$, we find that if $P \in V_0$, $T^{\text{proj}}_{V,P} \cap U_0 = T^{\text{aff}}_{V_0,P}$.

Proposition. The set of singular points on a nonempty irreducible projective hypersurface is a proper Zariski closed subset. In particular, the set of smooth points is dense.

Proof. The set of singular points is the intersection of V with $\bigcap_i \mathbb{V}\left(\frac{\partial F}{\partial X_i}\right)$, so is a closed subvariety of V. If $V \cap \bigcap_i \mathbb{V}\left(\frac{\partial F}{\partial X_i}\right) = V$, then by the Nullstellensatz, $\frac{\partial F}{\partial X_i} \in I^h(V)$. However, $I^h(V)$ is principal and generated by F. Since $\frac{\partial F}{\partial X_i}$ is homogeneous and of smaller degree, $\frac{\partial F}{\partial X_i} \mid F$ gives that $\frac{\partial F}{\partial X_i} = 0$. So F is a constant polynomial, giving $V = \mathbb{P}^n$ as it is nonempty, which has no singular points. $\square$

We can extend the definition of a tangent space to general varieties not generated by a single polynomial.

Definition. Let $V \subseteq \mathbb{A}^n$ be an affine variety, and let $P \in V$. Then the *tangent space* to V at P is

$$T_{V,P} = \left\{ \mathbf{v} \in \mathbb{C}^n \;\middle|\; \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i}(P) = 0 \text{ for all } f \in I(V) \right\} \subseteq \mathbb{C}^n$$

This is a vector subspace of $\mathbb{C}^n$.

Definition. Let $V \subseteq \mathbb{P}^n$ be a projective variety, and let $P \in V$. Suppose $V_j = V \cap \{X_j \neq 0\}$ is an affine piece containing P. Then the *tangent space* to V at P is $T_{V,P} = T_{V_j,P}$.

Note that this definition requires a choice of j; it is not clear that this is well-defined. This will be addressed by the following propositions.

Recall that $\mathbb{P}^n$ is covered by $U_0, \dots, U_n$, and $U_i \simeq \mathbb{A}^n$. Each point $P \in \mathbb{P}^n$ is contained in at least one of these U_i. If the index is unimportant, we will write $\mathbb{A}_n \subseteq \mathbb{P}^n$ rather than $U_i \subseteq \mathbb{P}^n$.

Let $V \subseteq \mathbb{P}^n, W \subseteq \mathbb{P}^m$ be irreducible varieties and $\varphi : V \dashrightarrow W$ be a rational map. Given $P \in \operatorname{dom} \varphi \subseteq V$ and $Q = \varphi(P) \subseteq W \cap \mathbb{A}^m$, we will now define $\mathrm{d}\varphi_P : T_{V,P} \dashrightarrow T_{W,P}$. Suppose φ is determined by $(F_0, \dots, F_m)$, where the F_i are homogeneous and of the same degree. By restricting to $\mathbb{A}^n$, we can write $\frac{F_j}{F_0}(1, X_1, \dots, X_n) = f_j \in \mathbb{C}(X_1, \dots, X_n)$. This gives rational functions $f_1, \dots, f_m$ on $V \cap \mathbb{A}^n$. The *derivative* of φ at P or *linearisation* of φ at P is defined by

$$\mathrm{d}\varphi_P(v) = \left(\sum_{i=1}^n v_i \frac{\partial f_j}{\partial X_i}(P) \right)_j$$

which is initially a function $T_{V,P} \to \mathbb{C}^m$. This can be thought of as an application of the matrix of derivatives of f at P to the vector v.

Proposition. (i) $\mathrm{d}\varphi_P(T_{V,P}) \subseteq T_{W,Q}$;
(ii) the linear map $\mathrm{d}\varphi_P$ depends only on φ and not the representatives;
(iii) if $\psi : W \dashrightarrow Z$ is rational with $\varphi(P) \in \operatorname{dom} \psi$, then $\mathrm{d}(\psi \circ \varphi)_P = \mathrm{d}\psi_{\varphi(P)} \circ \mathrm{d}\psi_P$;
(iv) if φ is birational and φ^{-1} is regular at $\varphi(P)$, then $\mathrm{d}\varphi_P$ is an isomorphism $T_{V,P} \simeq T_{W,Q}$.

Proof. Part (i). We use Y_j for coordinates in W. Replace V with V_0 and W with W_0. Let $g \in I(W)$, and consider its linearisation at Q. Applying the map $\varphi^\star$ on function fields, we obtain $\varphi^\star(g) = h = g(f_1, \dots, f_m) \in \mathbb{C}(V)$. Choose a representative in $\mathbb{C}(X)$, representing a rational function on V that is regular at P. This map vanishes when it is regular as $\varphi(\operatorname{dom}\varphi) \subseteq W$. By the chain rule,

$$\frac{\partial h}{\partial X_i}(P) = \sum_j \frac{\partial g}{\partial Y_j}(Q)\frac{\partial f_j}{\partial X_i}(P)$$

Hence, $v \in T_{V,P}$ gives $\mathrm{d}\varphi_P(v) \in T_{W,Q}$.

Part (ii). If $(F_0', \dots, F_m')$ is another representation of φ with corresponding rational functions $f_1', \dots, f_m' \in \mathbb{C}(V)$. Then $f_j - f_j'$ vanishes on V whenever it is defined, or equivalently, $f_j - f_j' = \frac{p_j}{q_j}$ where $p_j \in I(V)$ and $q_j(P) \neq 0$. Applying the quotient rule and the fact that $p_j \in I(V)$,

$$\frac{\partial(f_j - f_j')}{\partial X_i} = \frac{-1}{q_j(P)} = \frac{\partial p_j}{\partial X_i}(P) = 0$$

Hence, $v \in T_{V,P}$ gives $\sum_i v_i \frac{\partial(f_j - f_j')}{\partial X_i}(P) = 0$ as required.

Part (iii). Follows from the chain rule from multivariate calculus.

Part (iv). Immediate from (iii). □

Note that if $P \in U_i \cap U_j$, we have two different definitions of the tangent space $T_{V,P}$. Suppose that $V = \mathbb{P}^n$, then there is a birational map $p_{ij} : U_i \dashrightarrow U_j$ which is the identity on $U_i \cap U_j$. Part (iv) of the above proposition gives an isomorphism from T_{P,U_i} to T_{P,U_j} given by $\mathrm{d}p_{ij}$.

4.2 Smooth and singular points

Definition. Let V be an affine or projective variety. If V is irreducible, the *dimension* of V, written $\dim V$, is the minimum dimension of a tangent space for a point in V. If $P \in V$ and V is irreducible, we say P is a *smooth point* of V if $\dim T_{V,P} = \dim V$. Otherwise, P is a *singular point*. If V is reducible, we define $\dim V$ to be the maximum dimension of an irreducible component of V.

Theorem. Let V be a nonempty irreducible affine or projective variety. Then the set of smooth points of V is a nonempty open subset of V.

Proof. The fact that the set is nonempty is clear as the minimum dimension must be attained at a point. We can assume $V \subseteq \mathbb{A}^n$ is affine. If $P \in V$,

$$T_{V,P} = \left\{ \mathbf{v} \in \mathbb{C}^n \,\middle|\, \sum_{i=1}^n v_i \frac{\partial f_j}{\partial x_i}(P) = 0 \right\}$$

where f_j is some finite set of functions with $\mathbb{V}(\{f_j\}) = V$. Then

$$\dim T_{V,P} = n - \operatorname{rank} \frac{\partial f_j}{\partial X_i}(P)$$

For any $r \in \mathbb{N}$,

$$\left\{P \in V \mid \dim T_{V,P} \geq r\right\} = \left\{P \in V \,\middle|\, \operatorname{rank} \frac{\partial f_j}{\partial X_i}(P) \leq n - r\right\}$$

This is the subvariety given by the vanishing locus of the $(n-r+1) \times (n-r+1)$ minors of this matrix $\frac{\partial f_j}{\partial X_i}(P)$, which is closed. $\square$

Corollary. If V, W are irreducible and birational, then $\dim V = \dim W$.

4.3 Transcendental extensions

If $K \subseteq L$ are fields and $\alpha \in L$, we say that α is *transcendental* over K if it is not a solution to a nontrivial polynomial $f \in K[t]$. More generally, if $S \subseteq L$ is any set of elements, we say they are *algebraically independent* if they do not satisfy a nontrivial polynomial relation over K. A field extension $K/\mathbb{C}$ is a *pure transcendental extension* if K is generated by transcendental algebraically independent elements $x_1, \dots, x_n \in K$.

If V is an irreducible affine variety, recall that $\mathbb{C}(V) = FF\left({}^{\mathbb{C}[\mathbf{X}]}\!/\!_{I(V)}\right)$. If $V = \mathbb{P}^1$, $\mathbb{C}(V) \simeq \mathbb{C}(X)$.

Proposition. Let $K/\mathbb{C}$ be a finitely generated field extension. Then, there exists a pure transcendental subfield $K_0 = \mathbb{C}(x_1, \dots, x_m) \subseteq K$ such that K/K_0 is finite (and hence algebraic). Moreover, $K = K_0(y)$ for some $y \in K$.

Proof. The final statement follows from the primitive element theorem from Part II Galois Theory. We now prove the first part. K is finitely generated, so let $x_1, \dots x_n$ generate K. There is a maximal algebraically independent subset which after relabelling is given by $\{x_1, \dots, x_m\}$ for $m \leq n$. Then $x_{m+1}, \dots, x_n$ are algebraic over $K_0 = \mathbb{C}(x_1, \dots, x_m)$. $\square$

Proposition. Let $K = \mathbb{C}(x_1, \dots, x_n)$, where $x_1, \dots, x_n$ are algebraically independent. Let x_{n+1} be algebraic over K. Define

$$I = \{g \in \mathbb{C}[X_1, \dots, X_{n+1}] \mid g(x_1, \dots, x_n, x_{n+1}) = 0\}$$

Then I is a principal ideal generated by an irreducible element $f \in \mathbb{C}[\mathbf{X}]$. Moreover, if f contains the variable X_i, then $\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, x_{n+1}\}$ is algebraically independent.

Proof. As $x_1, \dots, x_n$ are algebraically independent, the subring $R = \mathbb{C}[x_1, \dots, x_n] \subseteq K$ is isomorphic to the polynomial ring $\mathbb{C}[X_1, \dots, X_n]$. $\mathbb{C}[X_1, \dots, X_n]$ is a unique factorisation domain. There exist polynomials $g \in K[T]$ where x_{n+1} is a root, as it is algebraic. Since $K[T]$ is a principal ideal domain, the ideal of such polynomials is principal, and generated by a unique monic polynomial $h(t)$, called the minimal polynomial of x_{n+1}. The minimal polynomial is irreducible.

Let b be the least common multiple of the denominators in $h(t)$, so $b \in R$. By Gauss' lemma, $f = bh$ is irreducible in $R[T]$. By the isomorphism $R \simeq \mathbb{C}[X_1, \dots, X_n]$, we can think of f as an element of $\mathbb{C}[X_1, \dots, X_{n+1}]$.

We show that f generates I. Suppose $g \in \mathbb{C}[\mathbf{X}]$ such that $g(x_1, \dots, x_{n+1}) = 0$. In $K[T]$, $g(x_1, \dots, x_n, T)$ is divisible by $f(x_1, \dots, x_n)$. By Gauss' lemma, $f \mid g$ in $\mathbb{C}[\mathbf{X}]$. Hence f generates I as required. The last part is left as an exercise. $\square$

Corollary. Let V be any irreducible variety. Then V is birational to a hypersurface.

Proof. Let K be the function field of V. By the above discussion, we can find elements that generate K that are given by $x_1, \dots, x_{n+1}$ where $x_1, \dots, x_n$ are algebraically independent and x_{n+1} is algebraic over $\mathbb{C}(x_1, \dots, x_n)$. By the previous proposition, $K \supseteq \mathbb{C}[x_1, \dots, x_{n+1}] = {}^{\mathbb{C}[X_1, \dots, X_{n+1}]}\!/_{(f)}$. We take the hypersurface $\mathbb{V}(f) \subseteq \mathbb{A}^{n+1}$. $\square$

We have shown above that birational varieties have the same dimension. We therefore have the following corollary.

Corollary. Let W be an irreducible variety, and let $V = \mathbb{V}(f) \subseteq \mathbb{A}^n$ be an affine hypersurface birational to W, where f is non-constant. Then the dimension of W is equal to $n - 1$.

In the language of field theory, the dimension of W is the transcendence degree of the field $\mathbb{C}(W)$.

5 Algebraic curves

5.1 Curves

Definition. A *curve* is a variety of dimension 1.

For our purposes, a curve is taken to mean a smooth irreducible projective variety of dimension 1. By convention, a curve C implicitly has an expression as $\mathbb{V}(I) \subseteq \mathbb{P}^n$, but this ambient space will not play an important role.

Example. Let $f_d \in \mathbb{C}[X, Y, Z]$ be homogeneous of degree d. For almost all choices of coefficients, $\mathbb{V}(f_d)$ is a (smooth irreducible projective) curve. We will show that for $d, d' \geq 2$, $\mathbb{V}(f_d)$ and $\mathbb{V}(f_{d'})$ are never isomorphic.

Proposition. Let C be a curve, and let $D \subsetneq C$ be a proper Zariski closed subset. Then D is a finite union of points.

Proof. It suffices to prove this for irreducible affine curves $V \subseteq \mathbb{A}^n$. Let $W \subsetneq V$ be a proper irreducible closed subset; we will show this is a single point. By the Nullstellensatz, there is a strict containment $I(V) \subsetneq I(W)$.

If $t \in \mathbb{C}[W] \setminus \mathbb{C}$, we can use this to produce an element $y \in \mathbb{C}[V]$ as follows. $\varphi : W \hookrightarrow V$ gives the pullback map $\varphi^\star : \mathbb{C}[V] \to \mathbb{C}[W]$ which is a surjection. Take any y such that $\varphi^\star(y) = t$.

We can also take $x \in \mathbb{C}[V]$ such that $\varphi^\star(x) = 0$, so $x \notin \mathbb{C}$. One can show that x, y are algebraically independent in $\mathbb{C}(V)$, as t is transcendental. This gives two algebraically independent elements of

$C(V)$, which has transcendence degree 1. So no such t can exist, so $\mathbb{C}[W] = \mathbb{C}$. Therefore W is a point. $\square$

Recall that if V is an irreducible variety, it has a coordinate ring (if it is affine), a function field, a local ring at each point, and the maximal ideal of functions vanishing at the given point in the local ring. These can be specialised in the case of curves. Note that if C is a smooth irreducible projective curve, there exists $t \in \mathbb{C}(V)$ such that ${}^{\mathbb{C}(V)}\!/_{\mathbb{C}(t)}$ is finite.

Theorem. Let P be a smooth point of an irreducible curve V. Then, the ideal $\mathfrak{m}_P \trianglelefteq \mathcal{O}_{V,P}$ is principal.

A generator π_P of $\mathfrak{m}_P$ is called a *local parameter*, a *local coordinate*, or a *uniformiser*.

Proof. We assume P lies in the affine patch V_0 of V. By changing coordinates, we can set $P = 0 \in \mathbb{A}^n$.

$$\begin{aligned}
\mathbb{C}[V_0] &= {}^{\mathbb{C}[X_1,\dots,X_n]}\!/_{I(V_0)} = \mathbb{C}[x_1,\dots,x_n]; \\
\mathcal{O}_P = \mathcal{O}_{V_0,P} &= \left\{ \frac{f}{g} \,\middle|\, f, g \in \mathbb{C}[V_0], g \notin (x_1,\dots,x_n) \right\} \\
\mathfrak{m}_P &= \left\{ \frac{f}{g} \,\middle|\, f \in (x_1,\dots,x_n), g \notin (x_1,\dots,x_n) \right\} = x_1\mathcal{O}_P + \dots + x_n\mathcal{O}_P \subseteq \mathcal{O}_P
\end{aligned}$$

where x_i is the image of X_i under the quotient map. More generally, if $J \trianglelefteq \mathcal{O}_P$ is any ideal, $\frac{f}{g} \in J$ if and only if $f \in J$. Therefore,

$$J = \left\{ \frac{f}{g} \,\middle|\, f \in J \cap \mathbb{C}[V_0], g \in \mathbb{C}[V_0], g(P) \neq 0 \right\}$$

In particular, J is finitely generated.

Since P is smooth, $T^{\mathrm{aff}}_{V_0,P}$ is a line, and by changing coordinates,

$$T_{V,P} = \{X_2 = X_3 = \dots = X_n = 0\}$$

We claim that x_1 generates $\mathfrak{m}_P$. Since $T_{V,P}$ is cut out by linearisations at $P = 0$ of elements in $I(V_0)$, there exist functions $f_2, \dots, f_n \in I(V_0)$ such that $f_j = X_j - h_j$ where h_j has no terms of degree less than 2. In $\mathcal{O}_P$,

$$x_j = h_j(x_1,\dots,x_n) \in (x_1^2, x_1x_2, \dots, x_n^2) = \mathfrak{m}_P^2$$

Thus, $\mathfrak{m}_P = \sum_{j=1}^n x_i\mathcal{O}_P = x_1\mathcal{O}_P + \mathfrak{m}_P^2$. The result that $\mathfrak{m}_P$ is generated by x_1 follows from Nakayama's lemma.

Lemma (Nakayama)**.** Let R be a ring, let M be a finitely generated R-module, and let $J \trianglelefteq R$ be an ideal. Then,

(i) if $JM = M$, there exists $r \in J$ such that $(1 + r)M = 0$; and
(ii) if $N \leq M$ is a submodule such that $JM + N = M$, then there exists $r \in J$ such that $(1 + r)M \subseteq N$.

Let

$$R = \mathcal{O}_L \supseteq J = \mathfrak{m}_P = M \supseteq N = (x_1)$$

and apply part (ii) of Nakayama's lemma to conclude. $\square$

Corollary. Let $V = \mathbb{V}(f) \subsetneq \mathbb{A}^2$ be an irreducible affine curve. Then, if $P \in V$ is a smooth point, the function $V \to \mathbb{C}$ defined by $Q \mapsto X(Q) - X(P)$ is a local parameter if and only if $\frac{\partial f}{\partial Y}(P) \neq 0$.

Proof. Use the proof of the above theorem. □

Corollary. Let P be a smooth point of a curve V. Then there exists a surjective group homomorphism $\nu_P : \mathbb{C}(V)^\star \to \mathbb{Z}$ called the *valuation* at P or *order of vanishing* at P, such that

(i) $\mathcal{O}_{V,P} = \{0\} \cup \{f \in \mathbb{C}(V)^\star \mid \nu_P(f) \geq 0\}$;

(ii) $\mathfrak{m}_p = \{0\} \cup \{f \in \mathbb{C}(V)^\star \mid \nu_P(f) > 0\}$;

(iii) if $f \in \mathbb{C}(V)^\star$, then for any local parameter π_P, we can write $f = \pi_P^{\nu_P(f)} u$ where $u \in \mathcal{O}_{V,P}^\star = \mathcal{O}_{V,P} \setminus \mathfrak{m}_P$.

We will 'filter' the ring $\mathcal{O}_{V,P}$ by ideals generated generated by powers π_P^k for $k \geq 0$.

Proof. We know that $\mathfrak{m}_P = (\pi_P)$, so $\mathfrak{m}_P^n = (\pi_P^n)$. Define $J = \bigcap_{n \geq 0} \mathfrak{m}_P^n$. Note that $J \trianglelefteq \mathcal{O}_{V,P}$ is a finitely generated ideal as we have seen in the previous proof, and moreover, $\mathfrak{m}_P J = \pi_P J = J$. By part (i) of Nakayama's lemma, it follows that $J = 0$. So only the zero function vanishes to infinite order.

For every $f \in \mathcal{O}_{V,P} \setminus \{0\}$, there exists a unique n such that $f \in \mathfrak{m}_P^n \setminus \mathfrak{m}_P^{n+1}$. Define $\nu_P(f) = n$ for this n. If $f \in \mathbb{C}(V) \setminus \mathcal{O}_{V,P} \setminus \{0\}$, we claim $f^{-1} \in \mathcal{O}_{V,P}$. Indeed, $f = \frac{g}{h}$ for $g, h \in \mathcal{O}_{V,P}$, so we can write $g = \pi_P^k u$ and $h = \pi_P^\ell u'$ where $k, l \geq 0$ and $u, u' \in \mathcal{O}_{V,P}^\star$. Since $f \notin \mathcal{O}_{V,P}$, it follows that $k < \ell$, so $f^{-1} \in \mathcal{O}_{V,P}$ as required. Given this, we can define $\nu_P(f) = -\nu_P(f^{-1})$ for such f.

As $\mathfrak{m}_P$ is a local ring, $\mathcal{O}_{V,P} \setminus \mathfrak{m}_P = \mathcal{O}_{V,P}^\star$, so every nonzero $f \in \mathbb{C}(V)$ is $\pi_P^{\nu_P(f)} u$ where $u \in \mathcal{O}_{V,P}^\star$, giving ν_P as desired. □

Example. Let $V = \mathbb{A}^1$ and $P = 0 \in \mathbb{A}^1$. Then

$$\mathcal{O}_{\mathbb{A}^1,0} = \left\{ \frac{f(t)}{g(t)} \,\middle|\, g(0) \neq 0 \right\}; \quad \mathfrak{m}_0 = \left\{ \frac{f(t)}{g(t)} \,\middle|\, f(0) = 0, g(0) \neq 0 \right\}$$

So $\mathfrak{m}_0$ is the set of $\frac{f(t)}{g(t)}$ where $t \mid f$. Then $\mathfrak{m}_0^k$ is the set of $\frac{f(t)}{g(t)}$ where $t^k \mid f$.

We can think of $\frac{f(t)}{g(t)}$ where $g(t) = a_0 + a_1 t + \dots + a_k t^k$ as $f(t)$ multiplied by the power series expansion of $g(t)^{-1}$ which has nonzero constant term. This product can be written as t^M multiplied by another power series with nonzero constant term. The valuation of f is $\nu_0\left(\frac{f}{g}\right) = M$.

Corollary. Let V be an irreducible curve and $f \in \mathbb{C}(V)$. If P is a smooth point, f or f^{-1} is regular at P.

Proof. f is regular at P if and only if $f \in \mathcal{O}_{V,P}$. The statement then follows by checking the sign of $\nu_P(f)$. □

Corollary. Let V be a smooth curve. Then any rational map $V \dashrightarrow \mathbb{P}^m$ is a morphism.

Proof. Reordering coordinates, we can assume the image of $\varphi : V \dashrightarrow \mathbb{P}^m$ is not contained in $\{X_0 = 0\}$. We write $\varphi = (G_0, \dots, G_m) = (1 : g_1 : \dots : g_m)$ where $g_j = \frac{G_J}{G_0} \in \mathbb{C}(V)$. If all $g_j \in \mathcal{O}_{V,P}$, the result holds. Otherwise, let $t = \min_j \{\nu_P(g_j)\}$, so $t < 0$. Note that $\min_j \{\nu_P(\pi_P^{-t} g_j)\} = 0$. Then $\varphi \sim (\pi_P^{-t} : \pi_P^{-t} g_1 : \dots : \pi_P^{-t} g_m)$ which is regular at P. □

Since every projective variety is contained in $\mathbb{P}^m$, any rational map from a curve to a projective variety is a morphism.

5.2 Maps between curves

Example. Let $C_d \subseteq \mathbb{P}^2$ be a smooth plane curve of degree d, so $C_d = \mathbb{V}(f)$ where f is homogeneous of degree d. Let $P \in \mathbb{P}^2$. Then, the projection from P, which is a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, automatically restricts to a morphism $C_d \to \mathbb{P}^1$. This morphism is surjective, and most points in $\mathbb{P}^1$ have a fibre of size d.

Proposition. Let $\varphi : V \to W$ be a non-constant morphism of irreducible (possibly singular) projective curves. Then, for all $Q \in W$, the fibre $\varphi^{-1}(Q)$ is finite. The map φ induces an inclusion $\varphi^\star : \mathbb{C}(W) \hookrightarrow \mathbb{C}(V)$ which makes $\mathbb{C}(V)$ a finite extension of $\mathbb{C}(W)$.

Proof. For the first statement, $\varphi^{-1}(Q)$ is Zariski closed in V, so is either V or a finite set of points. As φ is not constant, the fibre is a finite set of points. V is infinite, so by the first part, $\varphi(V)$ is infinite and therefore dense in W. Since φ is dominant, $\varphi^\star$ is defined. The map is automatically injective. Let $t \in \mathbb{C}(W) \setminus \mathbb{C}$ with $\varphi^\star(t) = x$. Since $\mathbb{C}(V)$ has transcendence degree 1 over $\mathbb{C}$, $\mathbb{C}(V)$ is finite over $\mathbb{C}(x)$, so also over $\mathbb{C}(W)$. □

Definition. Let $\varphi : V \to W$ be a non-constant morphism of curves. The *degree* of φ is the degree of the field extension $\mathbb{C}(V)/\varphi^\star\mathbb{C}(W)$.

Definition. Let $\varphi : V \to W$ be a non-constant morphism of curves, let $P \in V$ be a smooth point, and define $Q = \varphi(P)$. We define the *ramification degree* of φ at P by $e_P = e(\varphi, P) = \nu_P(\varphi^\star \pi_Q)$, where π_Q is a local coordinate at Q.

Example. Consider the morphism $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ defined by $z \mapsto z^d$ for some $d \geq 1$. On rings, this is given by $\varphi^\star : \mathbb{C}[Y] \to \mathbb{C}[X]$ with $\varphi^\star(Y) = X^d$. On function fields, this map satisfies $\varphi^\star\mathbb{C}(Y) = \mathbb{C}(X^d)$, a subfield of $\mathbb{C}(X)$. The degree of φ is d. Let $P = 0 \in \mathbb{A}^1$, so $Q = 0 \in \mathbb{A}^1$. A local parameter near Q is Y, and $\varphi^\star(Y) = X^d$. $\nu_0(X^d) = d$, so the ramification degree of φ at 0 is d.

Now suppose $P = 1$, $\varphi(P) = Q = 1$. The local coordinate at Q is $Y - 1$. We can find $\nu_P(\varphi^\star(Y-1)) = 1$, so the ramification degree of φ at 1 is 1. Note that $\varphi^{-1}(1)$ is the set of dth roots of unity, which is a set of d points $R_1, \dots, R_d$. $\nu_{R_i}(\varphi^\star(Y-1)) = 1$ for each i.

Theorem. Let $\varphi : V \to W$ be a non-constant morphism of irreducible projective curves.

(i) φ is surjective.

(ii) Suppose V, W are smooth. Then, for any $Q \in W$, $\deg \varphi = \sum_{P \in \varphi^{-1}(Q)} e_P$.

(iii) At all but finitely many points $P \in V$, $e_P = 1$.

Definition. A *quasi-projective variety U* is a Zariski-open subset of a projective variety $V \subseteq \mathbb{P}^n$.

Example. All projective varieties are quasi-projective. All affine varieties are also quasi-projective. Products of affine and projective varieties are quasi-projective, such as $\mathbb{P}^n \times \mathbb{A}^m$. Note that rational functions, rational maps, morphisms, irreducibility, function fields, local rings, and other algebraic geometric concepts are defined for quasi-projective varieties in the same way.

Proposition (fundamental theorem of elimination theory)**.** The projection map $\mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ is a Zariski closed map.

Preimages and images of closed sets are closed under this map.

Remark. Consider the map $\pi : \mathbb{A}^2 \to \mathbb{A}^1$ given by projection onto the x-axis. Observe that π is not a closed map, as $\mathbb{V}(XY - 1)$ has image $\mathbb{A}^1 \setminus \{0\}$, which is not closed.

Given this proposition, we prove the following result.

Proposition. Let $\varphi : V \to W$ be a morphism of quasi-projective varieties. Suppose that V is projective. Then φ is closed.

Proof. Factorise φ as $V \to \Gamma_\varphi \subseteq V \times W \to W$, where $\Gamma_\varphi = \{(x, \varphi(x)) \mid x \in V\}$ is the graph of φ. Note that Γ_φ is closed as it is the preimage of the diagonal $\varphi \times \mathrm{id} : V \times W \to W \times W$. The diagonal $W \subseteq W \times W$ is closed, even though $W \times W$ is not given the product topology. Now, $V \subseteq \mathbb{P}^n$ is a closed subset as it is a projective variety. Hence, it suffices to show that the projection map $\mathbb{P}^n \times W \to W$ is closed. Moreover, if W is covered by affine varieties $\{U_i\}$, it further suffices to show that $\mathbb{P}^n \times U_i \to U_i$ is closed for all i. Any quasi-projective variety is covered by affine varieties as required. Finally, each U_i is a closed subset of $\mathbb{A}^m$ for some m with its subspace topology. It therefore suffices to show $\mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ is closed, which is the fundamental theorem of elimination theory. ☐

We can now prove part (i) of the above theorem. Part (ii) is nonexaminable, and part (iii) will be shown later.

Corollary. Let $\varphi : V \to W$ be a non-constant map between irreducible projective curves. Then φ is surjective.

Proof. The image of φ is closed, so either a finite set of points or W itself. Since it is non-constant, φ is surjective. ☐

Corollary. Let V be a smooth projective irreducible curve, and let $f \in \mathbb{C}(V)^\star$. Then,

(i) if f is regular at all points $P \in V$, then $f \in \mathbb{C}^\star$ is a constant;

(ii) the set of $P \in V$ such that $\nu_P(f) \neq 0$ is finite, and $\sum_{P \in V} \nu_P(f) = 0$.

Proof. Part (i). Given f, consider the morphism $\varphi = (1 : f) : V \to \mathbb{P}^1$. φ is a morphism because C is smooth. We want to find zeroes and poles of f. $\varphi(P) = (1 : 0)$ if and only if $f(P) = 0$, and $\varphi(P) = (0 : 1)$ if and only if f is not regular at P. This means that if f is everywhere regular, φ is not surjective, so it is constant.

Part (ii). We can assume f is non-constant. Let t denote the rational function $\frac{X_1}{X_0}$ on $\mathbb{P}^1$. By the pullback, we obtain $\varphi^\star t \in \mathbb{C}(V)$ is exactly $\frac{f}{1} = f$. For convenience, $(1 : 0) \in \mathbb{P}^1$ will be denoted 0, and $(0 : 1) \in \mathbb{P}^1$ will be denoted ∞.

Observe that t is a local parameter at $0 \in \mathbb{P}^1$, so if $f(P) = 0$, $e_P = \nu_P(\varphi^\star t) = \nu_P(f)$. Similarly, $\frac{1}{t} = \frac{X_0}{X_1}$ is a local parameter at $\infty \in \mathbb{P}^1$, so if $f(P) = \infty$, we have $e_P = \nu_P(\varphi^\star \frac{1}{t}) = -\nu_P(f)$. Finally, if $f(P) \neq 0, \infty$, then $\nu_P(f) = 0$. By the previous theorem, $\deg \varphi = \sum_{\varphi(P)=0} \nu_P(f) = \sum_{\varphi(P)=\infty} -\nu_P(f)$, giving the desired result. □

Hence, there are no non-constant holomorphic functions.

5.3 Divisors

We will only consider smooth projective irreducible curves from now on. Let V be a curve. There is a natural inclusion from the space of functions defined everywhere on V (isomorphic to $\mathbb{C}$) to the field of rational functions on V. However, this field $\mathbb{C}(V)$ is very large and difficult to study directly. The goal of divisor theory is to organise $\mathbb{C}(V)$ into manageable (finite-dimensional) pieces.

Note that a rational function $f \in \mathbb{C}(V)$ determines an open subset $U \subseteq V$ on which f is well-defined as a function $U \to \mathbb{C}$. For instance, we could define $U = V \setminus \{x \mid \nu_P(f) < 0\}$, which is V with a finite set of points removed. One idea is to study functions $f \in \mathbb{C}(V)$ that are well-defined away from a fixed set $\{P_1, \dots, P_n\}$.

Definition. A *divisor* D on a curve V is a finite formal linear combination $\sum_{P \in V} n_P[P]$, or equivalently, an element of the free abelian group $\bigoplus_{P \in V} \mathbb{Z}[P]$. If $D = \sum_{P \in V} n_P[P]$, its *degree* is $\deg D = \sum_{P \in V} n_P \in \mathbb{Z}$.

Note that $\deg : \mathrm{Div}(V) \to \mathbb{Z}$ is a group homomorphism. The kernel of deg is denoted $\mathrm{Div}^0(V)$. If $D = \sum n_P[P]$, we write $\nu_P(D) = n_P$.

Definition. Let $D \in \mathrm{Div}(V)$. The space of rational functions on V with poles bounded by D is

$$L(D) = \{f \in \mathbb{C}(V) \mid f = 0 \text{ or } \forall P \in V,\ \nu_P(f) + \nu_P(D) \geq 0\}$$

For instance, if $\nu_P(D) > 0$, f is allowed to have a pole at P of order at most $\nu_P(D)$. If $\nu_P(D) < 0$, f is forced to have a zero at P of order at least $-\nu_P(D)$.

Definition. Let $f \in \mathbb{C}(V)^\star$. The *divisor of* f is $\operatorname{div}(f) = \sum_{P \in V} \nu_P(f)[P]$.

Divisors of rational functions must have degree 0. Divisors of the form $\operatorname{div}(f)$ are called *principal divisors*. The set $\operatorname{Prin}(V)$ is the set of divisors $D \in \operatorname{Div}(V)$ such that $D = \operatorname{div}(f)$ for some $f \in \mathbb{C}(V)^\star$, and this is a subgroup of $\operatorname{Div}^0(V)$, as $\operatorname{div}(f \cdot g) = \operatorname{div} f + \operatorname{div} g$.

The quotient ${}^{\operatorname{Div}(V)}\!/\!_{\operatorname{Prin}(V)}$ is noted $\operatorname{Pic}(V) = \operatorname{Cl}(V)$, and this is called the *Picard group* or *class group* of V. The Picard group and class group coincide for smooth varieties, but are different in the study of general varieties and schemes.

Divisors D, D' are called *linearly equivalent* if $D - D'$ is $\operatorname{div}(f)$ for some $f \in \mathbb{C}(V)^\star$, so D is equivalent to D' in $\operatorname{Pic}(V)$. We write $D \sim D'$.

Proposition. Every degree 0 divisor on $\mathbb{P}^1$ is principal.

Note that every principal divisor is degree 0 in general.

Proof. Identify $\mathbb{P}^1$ with $\mathbb{C} \cup \{\infty\}$, where $\mathbb{C} \hookrightarrow \{(1 : z) \mid z \in \mathbb{C}\}$. Then, $D = \sum_{a \in \mathbb{C}} n_a[a] + n_\infty[\infty]$. Note that $n_\infty = -\sum_{a \in \mathbb{C}} n_a$. Let $f = \prod_{a \in \mathbb{C}} (t-a)^{n_a}$. This has a zero of order n_a at a. Hence, $\operatorname{div} f = D$; clearly, $\nu_a(\operatorname{div} f) = n_a$ for $a \in \mathbb{C}$, and $\frac{1}{t-a}$ is a local coordinate at ∞ for all $a \in \mathbb{C}$ where $t = \frac{X_1}{X_0}$, then we can calculate explicitly $\nu_\infty(\operatorname{div} f) = n_\infty$. $\square$

It is not always the case that every degree 0 divisor on a curve V is principal and $\operatorname{Pic}(V)$ is nontrivial; this gives rise to the notion of genus.

Definition. Let $V \subseteq \mathbb{P}^n$ be a curve. Consider the hyperplane $\mathbb{V}(L) \subseteq \mathbb{P}^n$ where L is a homogeneous linear polynomial. Assume $V \nsubseteq \mathbb{V}(L)$. The *hyperplane section* of V by $\mathbb{V}(L)$ is $\operatorname{div} L = \sum_{P \in V} n_P[P]$, where if $X_i(P) \neq 0$, $n_P = \nu_P\left(\frac{L}{X_i}\right)$.

This is well-defined as $\nu_P\left(\frac{L}{X_i}\right) = \nu_P\left(\frac{L}{X_j}\right)$ for $X_i(p) \neq 0, X_j(P) \neq 0$, as $\frac{X_i}{X_j} \in \mathcal{O}_{V,P}^\star$ so $\nu_P\left(\frac{X_i}{X_j}\right) = 0$. Note that all n_P are nonnegative in this case.

Proposition. Let $V \subseteq \mathbb{P}^n$ be as above, and let L, L' be linear homogeneous polynomials, neither vanishing on V. Then there is an equality

$$\operatorname{div} L - \operatorname{div} L' = \operatorname{div}\left(\frac{L}{L'}\right)$$

In particular, $\operatorname{div} L - \operatorname{div} L'$ is principal, and $\deg \operatorname{div} L = \deg \operatorname{div} L'$.

Definition. Let $V \subseteq \mathbb{P}^n$ be a curve. Then the *degree* of V is $\deg \operatorname{div} L$ where $V \nsubseteq \mathbb{V}(L)$.

Remark. A line in $\mathbb{P}^2$ is degree 1. A conic is degree 2.

We can generalise these notions.

(i) If $\varphi : V \to \mathbb{P}^n$ is any non-constant morphism, and L is a linear form, we can similarly define $\operatorname{div} L$ by using $\sum_{P \in V} n_P[P]$ where $n_P = \nu_P\left(\frac{\varphi^\star L}{X_i}\right)$ where $X_i(P) \neq 0$. This generalises the case where φ is an inclusion. As before, we assume $\mathbb{V}(L)$ does not contain $\operatorname{Im}\varphi$. Note that this map need not be injective.

(ii) If G is homogeneous of degree $m \geq 1$ and $\varphi : V \to \mathbb{P}^n$, one can similarly define $\operatorname{div} G = \sum_{P \in V} n_P[P]$ where $n_P = \nu_P\left(\frac{\varphi^\star G}{X_i^m}\right)$ for any i such that $X_i(P) \neq 0$.

Theorem (weak form of Bézout's theorem)**.** Let $V, V' \subseteq \mathbb{P}^2$ be smooth projective irreducible curves of degrees m, n. Then if $V \neq V'$, the number of intersection points of V and V' is at most mn.

We have already shown that this is the case when V' is a line on an example sheet.

Proof. Suppose V, V' are cut out by $\mathbb{V}(F), \mathbb{V}(G)$ of degrees m, n. We claim that the degree of $\operatorname{div} G$ as a divisor on V is mn. We can replace G by any other homogeneous polynomial of degree m by the previous proposition as it gives a linearly equivalent divisor. Replace G with L^m for a homogeneous linear polynomial L. Now, $\mathbb{V}(L) \cap V$ has size at most $n = \deg V$, so $\deg \operatorname{div} \varphi^\star G = nm$ as required, since $\operatorname{div}(\varphi^\star G) = \sum_{P \in V \cap \mathbb{V}(G)} n_P[P]$ where $n_P > 0$ (note that if $n_P > 0$ then G vanishes at P). □

5.4 Function spaces from divisors

Definition. A divisor D is called *effective* if $D = \sum n_P[P]$ for $n_P \geq 0$.

Recall that
$$L(D) = \{f \in \mathbb{C}(V) \mid f = 0 \text{ or } \operatorname{div} f + D \geq 0 \text{ pointwise}\}$$
is equivalently the set of $f \in \mathbb{C}(V)$ such that $\operatorname{div} f + D$ is effective.

Proposition. The set $L(D)$ is a complex vector subspace of $\mathbb{C}(V)$.

Proof. $\nu_P(f+g) \geq \min\{\nu_P(f), \nu_P(g)\}$, hence sums of the form $f+g$ lie in $L(D)$ if $f, g \in L(D)$. Clearly $L(D)$ is closed under scalar multiplication. □

Definition. Denote $\ell(D) = \dim_{\mathbb{C}} L(D)$.

Example. Let ∞ denote the point $(0 : 1) \in \mathbb{P}^1$, and let $D = m[\infty]$ where $m \geq 0$. Writing $t = \frac{X_1}{X_0}$, $L(D)$ is spanned by $1, t, t^2, \dots, t^m$. Hence, $\ell(D) = m + 1$.

Proposition. Let D be a divisor on V. Then,
(i) If $\deg D < 0$, then $L(D) = 0$.
(ii) If $\deg D \geq 0$, then $\ell(D) \leq \deg D + 1$.
(iii) For any $P \in V$, $\ell(D) \leq \ell(D - P) + 1$.
In particular, $L(D)$ is always finite-dimensional.

Proof. *Part (i).* If $L(D) \neq 0$ then there exists $f \neq 0$ with $f \in L(D)$ such that $\operatorname{div} f + D \geq 0$. But taking degrees, $\deg \operatorname{div} f = 0$ hence $\deg D \geq 0$, a contradiction.

Part (iii). Let $n = \nu_P(D)$. Define $\mathrm{ev}_P : L(D) \to \mathbb{C}$ by $f \mapsto (\pi_P^n f)(P)$, intuitively extracting the first nonzero term of the power series defining f at P. The kernel of this homomorphism is $L(D-P)$.

Part (ii). This now follows from parts (i) and (iii). If $d = \deg D$, then $\ell(D) \leq \ell(D-(d+1)P)+d+1 = d+1$ where the latter equality holds as $\deg(D-(d+1)P) < 0$. □

Proposition. Let D, E be divisors on a curve V such that $D \sim E$, or equivalently, $D - E$ is principal. Then $L(D)$ and $L(E)$ are isomorphic as complex vector spaces. In particular, $\ell(D) = \ell(E)$.

Proof. If $D - E$ is principal, it can be written as $\operatorname{div}(g)$. Multiplication by g (respectively g^{-1}) gives a linear map (respectively its inverse) $L(D) \to L(E)$. □

6 Differentials

6.1 Differentials over fields

Differentials on curves will allow us to tackle some interesting questions.

(i) Given $D \in \mathrm{Div}(V)$, can we calculate (or bound) $\ell(D)$?

(ii) (Brill–Noether theory) For what integers r, d does a curve V admit a morphism $\varphi : V \to \mathbb{P}^r$ of degree d such that $\operatorname{Im} V$ is not contained in a hyperplane?

(iii) (Hurwitz problem) When does there exist a morphism $V \to W$ of smooth projective curves?

Definition. Let $K/\mathbb{C}$ be a field extension. The *space of differentials*, written $\Omega_{K/\mathbb{C}}$, is the quotient vector space ${}^{M}\!/_{N}$ where M is the K-vector space spanned by symbols δx where $x \in K$, and N is the subspace of M generated by

$$\delta(x+y) - \delta(x) - \delta(y); \quad \delta(xy) - x\delta(y) - y\delta(x); \quad \delta(a)$$

where $x, y \in K, a \in \mathbb{C}$. Given $x \in K$, we define $\mathrm{d}x = \delta x + N \in \Omega_{K/\mathbb{C}}$. The *exterior derivative* is the $\mathbb{C}$-linear map $\mathrm{d} : K \to \Omega_{K/\mathbb{C}}$ mapping x to $\mathrm{d}x$.

Remark. More generally, if $\varphi : A \to B$ is a ring homomorphism, we could have defined $\Omega_\varphi = \Omega_{B/A}$ as a B-module as above.

Definition. Let U be a K-vector space. A $\mathbb{C}$-linear transformation $D : K \to U$ is called a *derivation* if $D(xy) = xD(y) + yD(x)$.

Example. The map $\mathrm{d} : K \to \Omega_{K/\mathbb{C}}$ is a derivation. The map $\frac{\mathrm{d}}{\mathrm{d}x} : \mathbb{C}(X) \to \mathbb{C}(X)$ is a derivation.

Lemma (universal property). Let U be a K-vector space A map $D\colon K \to U$ is a derivation if and only if there is a K-linear map $\lambda\colon \Omega_{K/\mathbb{C}} \to U$ such that $\lambda(\mathrm{d}x) = D(x)$ for all $x \in K$.

$$\begin{tikzcd} & K \arrow[dl, "d"'] \arrow[dd, "D"] \\ \Omega_{K/\mathbb{C}} \arrow[dr, dashed, "\lambda"'] & \\ & U \end{tikzcd}$$

The proof is very simple and omitted. Intuitively, $\mathrm{d}\colon K \to \Omega_{K/\mathbb{C}}$ is the 'best possible' derivation.

Remark. For any derivation D, if $y \in K$ and $y \neq 0$, $D(x) = D\left(y \cdot \frac{x}{y}\right) = yD\left(\frac{x}{y}\right) + \frac{x}{y}D(y)$, giving the quotient rule.

$$D\left(\frac{x}{y}\right) = \frac{yDx - xDy}{y^2}$$

Lemma. (i) Let $f = \frac{h}{g} \in \mathbb{C}(X_1, \dots, X_n)$ and write $y = f(x_1, \dots, x_n)$ for $x_1, \dots, x_n \in K$. Then

$$\mathrm{d}y = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(x_1, \dots, x_n)\, \mathrm{d}x_i$$

(ii) If $K = \mathbb{C}(x_1, \dots, x_n)$ for $x_i \in K$, then $\Omega_{K/\mathbb{C}}$ is spanned by $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ as a K-vector space.

Proof. Part (i) follows from the rules of calculus for $\mathrm{d}(xy)$, $\mathrm{d}\left(\frac{x}{y}\right)$ and $\mathbb{C}$-linearity. Part (ii) is immediate from part (i). $\square$

We have obtained divisors in two different ways: from rational functions, and from hyperplane sections of $V \to \mathbb{P}^r$. We will do the reverse, we will first construct divisors, and then use them to build maps $V \to \mathbb{P}^r$. Differentials are another way to construct divisors.

From now, we will write Ω_K for $\Omega_{K/\mathbb{C}}$.

Theorem. Let $K/\mathbb{C}(t)$ be finite, with t transcendental over $\mathbb{C}$. Then Ω_K is one-dimensional as a K-vector space, and is spanned by $\mathrm{d}t$.

Proof. First, suppose $K = \mathbb{C}(t)$, the function field of $\mathbb{P}^1$. By the lemma above, Ω_K is spanned by $\mathrm{d}t$. We need to show that Ω_K is nonzero, then it is clearly one-dimensional. By the universal property, it suffices to exhibit a single nonzero derivation on K. The function $\frac{\mathrm{d}}{\mathrm{d}t}$ is one such derivation.

Now suppose $K \neq \mathbb{C}(t)$. Write $K_0 = \mathbb{C}(t)$, so $K = \mathbb{C}(t, \alpha) = K_0(\alpha)$ for $\alpha \in K \setminus K_0$ algebraic over K_0. Let $h(t) \in K_0[X]$ be the minimal polynomial of α. By minimality of h, $h'(\alpha) \neq 0$ as it does not have a double root. By the previous lemma, $\mathrm{d}t, \mathrm{d}\alpha$ span Ω_K as a K-vector space.

If $f \in K_0[X]$, write $D_t f$ for $\frac{\partial f}{\partial t}$, by t-differentiating the coefficients. The lemma gives $0 = \mathrm{d}(h(\alpha)) = D_t h(\alpha)\,\mathrm{d}t + h'(\alpha)\,\mathrm{d}\alpha$. Hence Ω_K is spanned by $\mathrm{d}t$, so it suffices to show Ω_K is nonzero. As in the first part, it suffices to exhibit a single nonzero derivation on K.

First, define $D : K_0[X] \to K$ by $D(f) = D_t f$ if $f \in K_0$, $D(X) = \frac{-(D_t h)(\alpha)}{h'(\alpha)}$, and $D(X^n) = n\alpha^{n-1}D(X)$. One can check that the ideal $hK_0[X]$ is mapped to zero under D. This exhibits a nonzero derivation as required. □

6.2 Rational differentials

Definition. Denote $\Omega_V = \Omega_{\mathbb{C}(V)/\mathbb{C}}$. Elements of Ω_V are called *rational differentials*. A differential $\omega \in \Omega_V$ is *regular* at a point $P \in V$ if ω can be expressed as $\sum_i f_i\,\mathrm{d}g_i$ where $f_i, g_i \in \mathcal{O}_{V,P}$. Write

$$\Omega_{V,P} = \{\omega \in \Omega_V \mid \omega \text{ regular at } P\} \subseteq \Omega_V$$

Note that $\Omega_{V,P}$ is not a vector subspace over $\mathbb{C}(V)$, since we can multiply by functions that are not regular. However, it is a module over $\mathcal{O}_{V,P}$.

Recall that $\mathcal{O}_{V,P}$ contains the maximal ideal $\mathfrak{m}_P$, which is principal, giving local coordinates. We can make a similar construction in the context of differentials.

Theorem. $\Omega_{V,P}$ is a free $\mathcal{O}_{V,P}$-module generated by $\mathrm{d}\pi_P$ where π_P is a local coordinate at P. In other words, $\Omega_{V,P} = \left\{f\,\mathrm{d}\pi_P \mid f \in \mathcal{O}_{V,P}\right\}$.

Remark. If π, π' are local coordinates at P, $\mathrm{d}\pi = u\,\mathrm{d}\pi'$ where $u \in \mathcal{O}_{V,P}^\star$. More generally, if $\omega \in \Omega_V$, then $\pi^j\omega$ is regular, so lies in $\Omega_{V,P}$, for sufficiently large k. Given this theorem, we can always write $\omega \in \Omega_V$ as $f\,\mathrm{d}\pi_P$ where π_P is a local coordinate at P and $f \in \mathbb{C}(V)$.

Definition. Let $\omega \in \Omega_V$ and $P \in V$. Define $\nu_P(\omega) = \nu_P(f)$ where $\omega = f\,\mathrm{d}\pi_P$ and π_P is a local coordinate at P.

Lemma. Let $\omega \in \Omega_V$ be a nonzero differential. Then, $\nu_P(\omega) \neq 0$ for only finitely many points P.

Proof. As $\nu_P(f\,\mathrm{d}g) = \nu_P(f) + \nu_P(\mathrm{d}g)$ and $\nu_P(f) = 0$ for all but finitely many points, it suffices to only prove this lemma for the case $\omega = \mathrm{d}g$. Moreover, as g must be non-constant as $\mathrm{d}g \neq 0$, we can assume that g is transcendental. hence, ${}^{\mathbb{C}(V)}\!/_{\mathbb{C}(g)}$ is a finite extension. Consider $(1 : g) : V \to \mathbb{P}^1$. By the finiteness theorem for rational functions, there are only finitely many $P \in V$ such that $g(P) = \infty$ or $e_P > 1$.

If P is a point where $e_P = 1$, so the function is unramified, $\varphi^\star(t - g(P))$ is a local coordinate at P. But $\varphi^\star(t - g(P))$ is $g - g(P)$, so $\nu_P(\mathrm{d}g) = 0$. □

Differentials provide another source of divisors.

Definition. Let $\omega \in \Omega_V$. Then $\operatorname{div} \omega = \sum_{P \in V} \nu_P(\omega)[P]$.

Proposition. Let ω, ω' be nonzero rational differentials on V. Then, $\operatorname{div} \omega - \operatorname{div} \omega'$ is principal.

Proof. Since Ω_V is one-dimensional over $\mathbb{C}(V)$, we can write $\omega = f\omega'$ where $f \in \mathbb{C}(V)$. It follows from the definitions that $\operatorname{div} \omega - \operatorname{div} \omega' = \operatorname{div} f$. $\square$

If ω is a nonzero differential, $\operatorname{div} \omega$ gives a well-defined element in $\operatorname{Pic}(V) = \operatorname{Cl}(V) = {}^{\operatorname{Div}(V)}\!/\!_{\operatorname{Prin}(V)}$. We say that $\operatorname{div} \omega$ is a *canonical divisor*, and its equivalence class is the *canonical class*, denoted K_V. Sometimes K_V is also simply called the canonical divisor.

We now prove the above theorem.

Proof. We want to check that $\mathrm{d}\pi_P$ generates the module $\Omega_{V,P}$ over $\mathcal{O}_{V,P}$. Clearly $\mathcal{O}_{V,P}\,\mathrm{d}\pi_P \subseteq \Omega_{V,P}$; we want to check that the converse holds. Given $f \in \mathcal{O}_{V,P}$, $f - f(P) \in \mathfrak{m}_P$. Hence, $f = f(P) + \pi_P g \in \mathcal{O}_{V,P}$ where $g \in \mathcal{O}_{V,P}$. By the Leibniz rule, $\mathrm{d}f = g\,\mathrm{d}\pi_P + \pi_P\,\mathrm{d}g \in \mathcal{O}_{V,P}\,\mathrm{d}\pi_P + \pi_P\Omega_{V,P}$. Assume that $\Omega_{V,P}$ is finitely generated. Observe that

$$\mathcal{O}_P\,\mathrm{d}\pi_P \subseteq \Omega_{V,P} \subseteq \mathcal{O}_P\,\mathrm{d}\pi_P + \pi_P\Omega_{V,P}$$

Apply Nakayama's lemma to $R = \mathcal{O}_{V,P}, J = \mathfrak{m}_P, M = \Omega_{V,P}, N = \mathcal{O}_{V,P}\,\mathrm{d}\pi_P$.

To show $\Omega_{V,P}$ is finitely generated, choose an affine patch $V_0 \subseteq V$ containing P. Then $C[V_0] = \mathbb{C}[x_1, \dots, x_n]$ where the x_i generate $\mathbb{C}[V_0]$. If $f \in \mathcal{O}_{V,P}$, we can write $f = \frac{g}{h}$ where g, h are polynomials and $h(P) \neq 0$. Thus

$$\mathrm{d}f = \sum_{i=1}^n \left(\frac{h\frac{\partial g}{\partial X_i} - g\frac{\partial h}{\partial X_i}}{h^2} \right)(x_1, \dots, x_n)\,\mathrm{d}x_i$$

But $h(P) \neq 0$, so $\mathrm{d}f$ is in the $\mathcal{O}_{V,P}$-span of $\{\mathrm{d}x_i\}$. $\square$

Example. Let $V = \mathbb{P}^1$, and let t be the coordinate on the standard $\mathbb{A}^1 \subseteq \mathbb{P}^1$. For any $a \in \mathbb{C}$, the rational function $(t - a)$ is a local coordinate. At infinity, $\frac{1}{t}$ is a local coordinate.

We now calculate $\operatorname{div} \mathrm{d}t$. We have $\nu_a(\mathrm{d}t) = \nu_a(\mathrm{d}(t - a)) = 0$ for all $a \in \mathbb{C}$. Note that $\mathrm{d}t = -t^2\,\mathrm{d}\left(\frac{1}{t}\right)$ so

$$\nu_\infty(\mathrm{d}t) = \nu_\infty\left(\frac{-1}{\left(\frac{1}{t}\right)^2}\,\mathrm{d}\left(\frac{1}{t}\right) \right) = -2$$

Hence $\operatorname{div} \mathrm{d}t = -2[\infty]$, so the degree is nonzero, hence this divisor is not principal.

Definition. Let V be a curve. The *genus* of V is $g(V) = \ell(K_V)$.

$L(K_V)$ is not well-defined, but $\ell(K_V)$ is. Note that if $V = \mathbb{P}^1$, then $\operatorname{div} \mathrm{d}t = -2[\infty]$, so $\ell(K_{\mathbb{P}^1}) = 0$, as there are no rational functions on $\mathbb{P}^1$ that vanish to order 2 at infinity, apart from the zero function.

6.3 Differentials on plane curves

We will study curves in $\mathbb{P}^2$.

Example (smooth plane cubics)**.** Consider $V = \mathbb{V}(F) \subseteq \mathbb{P}^2$ where $F = X_0X_2^2 - \prod_{i=1}^3(X_1 - \lambda_i X_0)$ with $\lambda_1, \lambda_2, \lambda_3$ distinct complex numbers. This curve is nonsingular. To calculate the genus, we take the following steps.

(i) We first use the affine equation $f(x,y) = y^2 - \prod_{i=1}^3(x - \lambda_i)$, and write $f(x,y) = y^2 - g(x,y)$. Differentiating, $2y\,\mathrm{d}y = g'(x)\,\mathrm{d}x$ is a nontrivial relation in Ω_V.

(ii) Using this relation, we choose a convenient differential $\omega \in \Omega_V$; in this case, we will choose $\omega = \frac{\mathrm{d}x}{y}$.

(iii) Calculate $\operatorname{div}\omega$ by using the fact that if $\frac{\partial f}{\partial y}(P)$ is nonzero, $x - x(P)$ is a local parameter, and if $\frac{\partial f}{\partial x}(P)$ is nonzero, $y - y(P)$ is a local parameter.

We find that $K_V = 0$. Hence, $g(V) = 1$ as $\ell(0) = 1$.

Theorem. Let V be a smooth plane cubic. Then $g(V) = 1$, and in particular, $V \not\simeq \mathbb{P}^1$.

Proof. Change coordinates into the example above. □

Theorem. Let $V = \mathbb{V}(F) \subseteq \mathbb{P}^2$ be a smooth projective plane curve of degree d. Then $K_V = (d-3)H$ where H is the divisor class associated to a hyperplane section of V.

Proof. First, we will select a differential $\omega \in \Omega_V$. Change coordinates such that $(0 : 1 : 0) \notin V$. Let $x = \frac{X_1}{X_0}, y = \frac{X_2}{X_0}$ be elements of $\mathbb{C}(V)$. Set $f(X,Y) = F(1,X,Y)$, so $f(x,y) = 0$ in $\mathbb{C}(V)$. Differentiating, $\frac{\partial f}{\partial X}(x,y)\,\mathrm{d}x + \frac{\partial f}{\partial Y}(x,y)\,\mathrm{d}y = 0$ is a relation in Ω_V. Choose

$$\omega = \frac{\mathrm{d}x}{\frac{\partial f}{\partial Y}(x,y)} = \frac{-\,\mathrm{d}y}{\frac{\partial f}{\partial X}(x,y)}$$

Then, we will calculate $\operatorname{div}\mathrm{d}\omega$ in this affine patch. If $\frac{\partial f}{\partial Y}(P) \neq 0$, then $x - x(P)$ is a local coordinate at P. Then, $\nu_P(\omega) = \nu_P\left(\frac{1}{\frac{\partial f}{\partial Y}}(x,y)\right) = 0$. Otherwise, $\frac{\partial f}{\partial X}(P) \neq 0$ by smoothness, so $y - y(P)$ is a local coordinate and $\nu_P(\omega) = 0$.

Since $(0 : 1 : 0) \notin V$, any point at infinity in V is not contained in $\{X_2 = 0\}$. The equation for V on the patch $\{X_2 \neq 0\}$ is $g(z,w) = 0$ where $z = \frac{X_0}{X_2} = \frac{1}{y}$ and $y = \frac{X_1}{X_2} = \frac{x}{y}$ and $g(Z,W) = F(Z,W,1)$ in $\mathbb{C}[Z,W]$. Select a different differential

$$\eta = \frac{\mathrm{d}z}{\frac{\partial g}{\partial W}(z,w)} = \frac{-\,\mathrm{d}w}{\{g\}Z(z,w)}$$

By the same argument as before, $\nu_P(\eta) = 0$ for all P in the patch $\{X_2 \neq 0\}$. Using $f(X,Y) = Y^d g\left(\frac{1}{X}, \frac{X}{Y}\right)$ and differentiating, we find $\omega = Z^{d-3}\eta$. If $X_2(P) \neq 0$, we calculate $\nu_P(\omega) = (d-3)\nu_P(z) + \nu_P(\eta) = (d-3)\nu_P(z)$. As $Z = \frac{X_0}{X_2}$, $\operatorname{div}\omega = (d-3)\operatorname{div} X_0$ as claimed. □

Proposition. If $f(x,y) = 0$ is an affine patch equation for a smooth projective plane curve, and $\deg f \geq 3$, then

$$\left\{ \frac{x^r y^s \,\mathrm{d}x}{\frac{\partial f}{\partial y}} \;\middle|\; 0 \leq r, s;\; r + s \leq d - 3 \right\}$$

is a basis for $L(K_V)$ for the representative of K_V given by $(d-3)H$ where H is the hyperplane at infinity.

The $\mathrm{d}x$ term can be considered a dummy symbol, meant to indicate that we think of the term as a differential.

Proof. The proof is non-examinable, and follows from the same argument as the proof of the previous theorem. □

Corollary. If $d, d' \geq 2$ are distinct integers, then smooth plane curves of degrees d, d' are never isomorphic.

Proof. $\deg K_V$ depends only on V up to isomorphism. □

In particular, there are infinitely many distinct curves up to isomorphism.

6.4 The Riemann–Roch theorem

Theorem. Let V be a smooth irreducible projective curve of genus g, and let D be a divisor on V. Let K_V be the canonical divisor class. Then,

$$\ell(D) - \ell(K_V - D) = \deg(D) - g + 1$$

The proof is beyond the scope of this course. This theorem is related to Stokes' theorem and the Gauss–Bonnet theorem.

Corollary. Let K be the canonical divisor on V. Then, $\deg(K) = 2g - 2$.

Note that $2g - 2 = -\chi(V)$, the negative of the Euler characteristic of V.

Proof. Let $D = K$ in the Riemann–Roch theorem, and use $\ell(0) = 1$. □

Corollary. Let V be a smooth projective plane curve of degree d. Then the genus is $g(V) = \frac{(d-1)(d-2)}{2}$.

Proof. We have seen that if $d = 1, 2$ then $V \simeq \mathbb{P}^1$. If $d \geq 3$, we have seen that K is linearly equivalent to $(d-3)H$ where H is a hyperplane section. But $\deg(H) = d$, hence the result follows from the Riemann–Roch theorem. □

Given a divisor D on V, calculating $\ell(D)$ is hard with the techniques discussed so far. However, the Riemann–Roch theorem can be used to compute this for most D.

Corollary. If $\deg(D) > 2g - 2$, then $\ell(D) = \deg(D) - g + 1$.

Proof. The divisor $K - D$ has negative degree, hence $\ell(K - D) = 0$. □

We can compare this to the case $V = \mathbb{P}^1$, where we saw by direct calculation that $\ell(D) = \deg(D) + 1$.

Corollary. Suppose $g(V) = 1$. Then if D is a divisor with $\deg(D) > 0$, then $\ell(D) = \deg(D)$.

Proof. $\ell(K - D) = \ell(-D) = 0$. □

Let V be a curve of genus 1, and fix $P_0 \in V$. Let $P, Q \in V$, then $P + Q - P_0$ is equivalent to a unique effective divisor of degree 1. So $P + Q - P_0$ is equivalent to R for a unique $R \in V$. Indeed, $\deg(P + Q - P_0) = 1$ hence $\ell(P + Q - P_0) = 1$, so there exists a function $f \in \mathbb{C}(V)$ such that $(P + Q - P_0) + \operatorname{div}(f)$ is effective, and hence equal to a point R. It is unique as $\ell(P + Q - P_0) = 1$, and scalar multiples of f give the same divisor.

In other words, given $E = (V, P_0)$ as above, we can define $P +_E Q = R$ using the preceding notation. The pair (V, P_0) where $g(V) = 1, P_0 \in V$ is called an *elliptic curve*. Topologically, such V in the Euclidean topology are homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$; the group law defined by $+_E$ and that defined on $\mathbb{S}^1 \times \mathbb{S}^1$ in fact coincide.

Theorem. The operation $+_E$ gives E the structure of an abelian group with identity P_0. Moreover, the map $E \to \operatorname{Cl}^0(E) = \operatorname{Cl}^0(V)$ defined by $P \mapsto [P - P_0]$ is an isomorphism of groups.

Proof. Let $\beta(P) = [P - P_0] \in \operatorname{Cl}^0(E) = {}^{\operatorname{Div}^0(E)}\!/_{\operatorname{Prin}(E)}$. First, we show injectivity. Suppose $\beta(P) = \beta(Q)$, so $P - P_0 \sim Q - P_0$, where $\sim$ denotes linear equivalence. Hence $P \sim Q$. However, $\ell(P) = 1$ by the Riemann–Roch theorem, so $P = Q$.

Now, we show surjectivity. Suppose D has degree 0. We want to show D is equivalent to $[P - P_0]$ for some P. Since the degree of $D + P_0$ is 1, $\ell(D + P_0) = 1$ by Riemann–Roch. Hence there exists $P \in V$ such that $D + P_0 \sim P$. So $D = \beta(P)$ as required.

Hence β is a bijection of sets, so it remains to check that β is a homomorphism; this is straightforward. $\square$

Theorem. Let $E = (V, P_0)$ be the elliptic curve given by $\mathbb{V}(F)$ where $F = X_0X_2^2 - \prod_{i=1}^{3}(X_1 - \lambda_i X_0)$. Choose $P_0 = (0 : 0 : 1)$. Then, $P +_E Q +_E R = 0_E$ if and only if P, Q, R are collinear in $\mathbb{P}^2$.

The proof is nonexaminable.

Given a morphism $\varphi : V \to W$ of curves, we wish to understand the relation between $g(V)$ and $g(W)$. Let $\omega = f\,\mathrm{d}t \in \Omega_W$, where $\mathbb{C}(W)/\mathbb{C}(t)$ is finite. Since $\mathbb{C}(V)/\mathbb{C}(t)$ is finite, Ω_V is generated by $\mathrm{d}\varphi^\star t$. Define the pullback $\Omega_W \to \Omega_V$ by $\varphi^\star\omega = \varphi^\star f\,\mathrm{d}\varphi^\star t$. Let P be a point on V, and $Q = \varphi(P)$. We compare $\nu_P(\varphi^\star\omega)$ and $\nu_Q(\omega)$.

Lemma. Let π_P, π_Q be local parameters at P, Q. Let e_P be the ramification degree at P, so $\varphi^\star(\pi_Q) = u\pi_P^{e_P}$ where u is a unit in $\mathcal{O}_{V,P}$. Then, $\nu_P(\varphi^\star(\mathrm{d}\pi_Q)) = e_P - 1$. More generally, $\nu_P(\varphi^\star\omega) = e_P\nu_Q(\omega) + e_P - 1$.

This can be thought of as a generalisation of the rule $\frac{\mathrm{d}}{\mathrm{d}x}\{x^n\} = nx^{n-1}$.

Proof. For the first part, we have that $\varphi^\star(\pi_Q) = u\pi_P^{e_P}$, so differentiating and taking valuation gives the desired result. For a general ω, we can write $\omega = u\pi_Q^m\,\mathrm{d}\pi_Q$ where u is a unit in $\mathcal{O}_{V,P}$ as $\Omega_{W,Q}$ is a free module generated by $\mathrm{d}\pi_Q$. Then, we can apply $\varphi^\star$ and proceed as in the first part. $\square$

Theorem (Riemann–Hurwitz)**.** Let $\varphi : V \to W$ be as above. Let $n = \deg\varphi$, $n \neq 0$. Then

$$2g(V) - 2 = n(2g(W) - 2) + \sum_{P \in V}(e_P - 1)$$

where e_P is the ramification of φ at P.

Note that the correction term $\sum_{P\in V}(e_P - 1)$ is nonnegative.

Proof. Let $\omega \in \Omega_W$ be nonzero. Then, by the Riemann–Roch theorem, and the previous lemma,

$$\begin{aligned} 2g(V) - 2 &= \deg(\operatorname{div}(\varphi^\star \omega)) \\ &= \sum_{P \in V} \nu_P(\varphi^\star \omega) \\ &= \sum_{Q \in W} \sum_{P \in \varphi^{-1}(Q)} \nu_P(\varphi^\star \omega) \\ &= \sum_{Q \in W} \sum_{P \in \varphi^{-1}(Q)} (e_P \nu_Q(\omega) + e_P - 1) \\ &= \sum_{Q \in W} \left(n\nu_Q(\omega) + \sum_{P \in \varphi^{-1}(Q)} (e_P - 1) \right) \\ &= n \deg(\operatorname{div}(\omega)) + \sum_{P \in V} (e_P - 1) \\ &= n(2g(W) - 2) + \sum_{P \in V} (e_P - 1) \end{aligned}$$

□

Corollary. Let V, W be curves with $g(V) < g(W)$. Then any rational map $V \dashrightarrow W$ is constant.

Proof. Any rational map of this form is a morphism, then apply the Riemann–Hurwitz theorem. □

For example, there is no map $\mathbb{P}^1 \to V$ for $g(V) \geq 1$.

6.5 Equations for curves using Riemann–Roch

Let $V \subseteq \mathbb{P}^n$ be a curve not contained in any hyperplane; this can be done without loss of generality by iteratively reducing n. Let $D = \operatorname{div}(X_0)$ be the hyperplane section. Let $G \in \mathbb{C}[\mathbf{X}]$ be a homogeneous linear polynomial. Then $f = \frac{G}{X_0} \in \mathbb{C}(V)^\star$. Observe that $\operatorname{div} f + D = \operatorname{div} G$ is effective. Hence $f \in L(D)$.

We thus obtain an injective linear map from the space of linear homogeneous polynomials in $\mathbb{C}[\mathbf{X}]$ into $L(D)$ defined by $G \mapsto \frac{G}{X_0}$. This is injective because V is not contained inside a hyperplane. We make the following observations.

(i) For any point $P \in V$, there exist linear homogeneous polynomials F, G such that $F(P) \neq 0$ and $G(P) = 0$.

(ii) If P is a smooth point and L is the tangent line in $\mathbb{P}^n$, we can find a linear homogeneous polynomial F such that $F(P) = 0$ but F does not vanish on all of L.

Under this injection, we obtain the following condition. We say that a divisor D on V satisfies condition $(\star)$ if for every $P, Q \in V$ not necessarily distinct, we have $\ell(D - P - Q) = \ell(D) - 2$.

Definition. Let V be a curve, and let D a divisor with $\ell(D) = n + 1 \geq 2$. Let $\{f_0, \dots, f_n\}$ be a basis for $L(D)$. The *morphism associated to D* is $\varphi_D : V \to \mathbb{P}^n$ given by $(f_0 : \dots : f_n)$.

We say that φ_D is an *embedding* if it is an isomorphism onto its image.

Theorem. The morphism φ_D associated to D is an embedding if and only if D satisfies condition $(\star)$.

The proof is omitted.

Corollary. Suppose D is a divisor with $\deg D > 2g$. Then φ_D is an embedding.

Proof. By Riemann–Roch, D satisfies $(\star)$. □

Corollary. Every curve of genus g can be embedded in $\mathbb{P}^m$ for some m depending only on g.

Proof. If $g \geq 3$, take $[D] = 2K_V$. If $g = 2$, take $[D] = 3K_V$. If $g = 1$, take $[D] = 3[P_0]$ for some $P_0 \in V$. In any case, $\deg D > 2g$. □

Definition. A curve V of genus $g(V) \geq 2$ is called *hyperelliptic* if there exists a degree 2 morphism $V \to \mathbb{P}^1$.

The following theorem is on the last example sheet.

Theorem. A curve of genus g is hyperelliptic if and only if there exists a divisor D such that $\deg D = 2$ and $\ell(D) = 2$.

Theorem. Let V be a curve of genus $g(V) \geq 2$ that is not hyperelliptic. Then, the morphism $\varphi_{K_V} : V \to \mathbb{P}^{g-1}$ is an embedding.

Proof. Suppose that φ_K is not an embedding. Then K violates $(\star)$, so there exist points $P, Q \in V$ such that $\ell(K - P - Q) \geq g - 1$. Then by Riemann–Roch, $D = P + Q$ has $\ell(D) \geq 2$. But this is the maximal value by the above inequalities, so the result follows. □