Complex Analysis

Cambridge University Mathematical Tripos: Part IB

21st May 2024

Contents

1 Differentiation

1.1 Basic notions

We use the following definitions.

- The complex plane is denoted $\mathbb{C}$.
- The complex conjugate of a complex number z is denoted $\overline{z}$.
- The modulus is denoted $|z|$.
- The function $d(z,w) = |z - w|$ is a metric on $\mathbb{C}$. All topological notions will be with respect to this metric.
- We define the disc $D(a,r) = \{z \in \mathbb{C} : |z - a| < r\}$ to be the open ball with centre a and radius r.
- A subset $U \subset \mathbb{C}$ is said to be open if it is open with respect to the above metric. In particular, by identifying $\mathbb{C}$ with $\mathbb{R}^2$, we can see that $U \subset \mathbb{C}$ is open if and only if $U \subset \mathbb{R}^2$ is open with respect to the Euclidean metric.

The course concerns itself with complex-valued functions of a single complex variable. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ allows us to construct $f(z) = u(x,y) + iv(x,y)$, where u, v are real-valued functions. We can denote these parts by $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$.

1.2 Continuity and differentiability

The definition of continuity is carried over from metric spaces. That is, $f : A \to \mathbb{C}$ is continuous at a point $w \in A$ if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall z \in A, |z - w| < \delta \implies |f(z) - f(w)| < \varepsilon$$

Equivalently, the limit $\lim_{z \to w} f(z)$ exists and takes the value $f(w)$. We can easily check that f is continuous at $w = c + id \in A$ if and only if u, v are continuous at (c,d) with respect to the Euclidean metric on $A \subset \mathbb{R}^2$.

Definition. Let $f : U \to \mathbb{C}$, where U is open in $\mathbb{C}$.

(i) f is *differentiable* at $w \in U$ if the limit

$$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w}$$

exists, and its value is complex. We say that $f'(w)$ is the derivative of f at w.

(ii) f is *holomorphic* at $w \in U$ if there exists $\varepsilon > 0$ such that $D(w,\varepsilon) \subset U$ and f is differentiable at every point in $D(w,\varepsilon)$.

(iii) f is holomorphic in U if f is holomorphic at every point in U, or equivalently, f is differentiable everywhere.

Differentiation of composite functions, sums, products and quotients can be computed in the complex case exactly as they are in the real case.

Example. Polynomials $p(z) \sum_{j=0}^{n} a_j z^j$ for complex coefficients a_j are holomorphic on $\mathbb{C}$. Further, if p, q are polynomials, $\frac{p}{q}$ is holomorphic on $\mathbb{C} \setminus \{z : q(z) = 0\}$.

Remark. The differentiability of f at a point $c + id$ is not equivalent to the differentiability of u, v at (c, d). $u: U \to \mathbb{R}$ is differentiable at $(c, d) \in U$ if there is a 'good' affine approximation of u at (c, d); there exists a linear transformation $L: \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{(x,y)\to(c,d)} \frac{u(x, y) - (u(c, d) - L(x - c, y - d))}{\sqrt{(x - c)^2 + (y - d)^2}} = 0$$

If u is differentiable at (c, d), then L is uniquely defined, and can be denoted $L = Du(c, d)$. L is given by the partial derivatives of u, which are

$$L(x, y) = \left(\frac{\partial u}{\partial x}(c, d)\right)x + \left(\frac{\partial u}{\partial y}(c, d)\right)y$$

This seems to imply that the differentiability of f requires more than the differentiability of u, v.

1.3 Cauchy–Riemann equations

Theorem. $f = u + iv: U \to \mathbb{C}$ is differentiable at $w = c + id \in U$ if and only if $u, v: U \to \mathbb{R}$ are differentiable at $(c, d) \in U$ and u, v satisfy the Cauchy–Riemann equations at (c, d), which are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

If f is differentiable at $w = c + id$, then

$$f'(w) = \frac{\partial u}{\partial x}(c, d) + i\frac{\partial v}{\partial x}(c, d)$$

and other expressions, which follow directly from the Cauchy–Riemann equations.

Proof. All of the following statements will be bi-implications. Suppose f is differentiable at w with $f'(w) = p + iq$, so

$$\lim_{z\to w} \frac{f(z) - f(w)}{z - w} = p + iq$$

$$\lim_{z\to w} \frac{f(z) - f(w) - (z - w)(p + iq)}{|z - w|} = 0$$

By separating real and imaginary parts, writing $w = c + id$ we have

$$\lim_{(x,y)\to(c,d)} \frac{u(x, y) - u(c, d) - p(x - c) + q(y - d)}{\sqrt{(x - c)^2 + (y - d)^2}} = 0$$

$$\lim_{(x,y)\to(c,d)} \frac{v(x, y) - v(c, d) - q(x - c) - p(y - d)}{\sqrt{(x - c)^2 + (y - d)^2}} = 0$$

Thus, u is differentiable at (c, d) with $Du(c, d)(x, y) = px - qy$ and v is differentiable at (c, d) with $Dv(c, d)(x, y) = qx + py$.

$$u_x(c, d) = v_y(c, d) = p; \quad -u_y(c, d) = v_x(c, d) = q$$

Hence the Cauchy–Riemann equations hold at (c, d). We also find that if f is differentiable at w, we have $f'(w) = u_x(c, d) + iv_x(c, d)$. □

Remark. If u, v simply satisfy the Cauchy–Riemann equations alone, that does not imply differentiability of f. u, v must also be differentiable.

Remark. If we simply want to show that the differentiability of f implies that the Cauchy–Riemann equations hold, we can proceed in a simpler way. For $t \in \mathbb{R}$,

$$f'(w) = \lim_{t \to 0} \left(\frac{u(c+t,d) - u(c,d)}{t} + i \frac{v(c+t,d) - v(c,d)}{t} \right)$$

Hence the real part and the complex part both exist, so $u_x(c,d)$ and $v_x(c,d)$ exist, and $f'(w) = u_x(c,d) + iv_x(c,d)$. If we instead considered a perturbation along the imaginary axis, we find $f'(w) = v_y(c,d) - iu_y(c,d)$, giving the Cauchy–Riemann equations.

Example. The complex conjugate function $z \mapsto \overline{z}$ is not differentiable. Here, $u(x,y) = x$, and $v(x,y) = -y$, so the Cauchy–Riemann equations do not hold.

Corollary. If u, v have continuous partial derivatives at (c,d) and satisfy the Cauchy–Riemann equations at this point, then f is differentiable at $c + id$. In particular, if u, v are C^1 functions on U (i.e. have continuous partial derivatives in U) satisfying the Cauchy–Riemann equations everywhere, then f is holomorphic (in U).

Proof. If u, v have continuous partial derivatives then u, v are differentiable at (c,d) by Analysis and Topology. □

1.4 Curves and path-connectedness

Definition. A *curve* is a continuous function $\gamma \colon [a,b] \to \mathbb{C}$, where $a, b \in \mathbb{R}$. γ is a C^1 curve if γ' exists and is continuous on $[a,b]$. An open set $U \subset \mathbb{C}$ is *path-connected* if for any two points $z, w \in U$, there exists $\gamma \colon [0,1] \to U$ such that $\gamma(0) = z$ and $\gamma(1) = w$. A *domain* is a non-empty, open, path-connected subset of $\mathbb{C}$.

Corollary. Let U be a domain. Let $f \colon U \to \mathbb{C}$ be a holomorphic function with derivative zero everywhere. Then f is constant on U.

Proof. By the Cauchy–Riemann equations, $f' = 0$ implies that $Du = Dv = 0$ in U. By Analysis and Topology, the path-connectedness of U implies that u and v are constant functions. □

1.5 Power series

Recall the following theorem from IA Analysis.

Theorem. Let $(c_n)_{n=0}^{\infty}$ be a sequence of complex numbers. Then, the power series

$$\sum_{n=0}^{\infty} c_n (z-a)^n$$

has a unique *radius of convergence* $R \in [0, \infty]$ such that the power series converges absolutely for $|z-a| < R$ and diverges if $|z-a| > R$. Further, if $0 < r < R$, the series converges uniformly with respect to z on the compact disc $D(a,r)$.

Note that

$$R = \sup\left\{r \geq 0 : \lim_{n\to\infty} |c_n|r^n = 0\right\}; \quad \frac{1}{R} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$$

Theorem. Let the sequence (c_n) define a power series f centred around a with positive radius of convergence R. Then, the function $f : D(a,R) \to \mathbb{C}$ satisfies

(i) f is holomorphic on $D(a,R)$;

(ii) the term-by-term differentiated series $\sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$ also has radius of convergence equal to R, and this series is exactly the value of f';

(iii) f has derivatives of all orders on $D(a,R)$ and $c_n = \frac{f^{(n)}(a)}{n!}$;

(iv) if f vanishes on $D(a,\varepsilon)$ for any $\varepsilon > 0$, then $f \equiv 0$ on $D(a,R)$.

Proof. (i) Without loss of generality, let $a = 0$. $\sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$ has some radius of convergence R_1.

Let $z \in D(0,R)$ and choose ρ such that $|z| < \rho < R$. Then,

$$n|c_n||z|^{n-1} = n|c_n|\left|\frac{z}{\rho}\right|^{n-1} \rho^{n-1} \leq |c_n|\rho^{n-1}$$

for sufficiently large n, since $n\left|\frac{z}{\rho}\right|^{n-1} \to 0$ as $n \to \infty$. Since $\sum |c_n|\rho^n$ converges, we must have that $n|c_n||z|^{n-1}$ converges. Hence $R_1 \geq R$.

Now, since

$$|c_n||z|^n \leq n|c_n||z|^n = |z|\left(n|c_n||z|^{n-1}\right)$$

If $\sum n|c_n|z^{n-1}$ converges then so does $\sum |c_n||z|^n$. Hence $R_1 \leq R$. This leads us to conclude $R_1 = R$.

(ii) Let $z \in D(0,R)$. The statement that f' is the above differentiated power series at z is equivalent to continuity at z of the function

$$g : D(0,R) \to \mathbb{C}; \quad g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & w \neq z \\ \sum_{n=1}^{\infty} nc_n z^{n-1} & w = z \end{cases}$$

Substituting for f, we have $g(w) = \sum_{n=1}^{\infty} h_n(w)$ for $w \in D(0,R)$ where

$$h_n(w) = \begin{cases} \frac{c_n(w^n-z^n)}{w-z} & w \neq z \\ nc_n z^{n-1} & w = z \end{cases}$$

Note that h_n is continuous on $D(0,R)$. Further, note that

$$\frac{w^n - z^n}{w-z} = \sum_{j=0}^{n-1} z^j w^{n-1-j}$$

We have that for all r with $|z| < r < R$ and all $w \in D(0,r)$, $|h_n|(w) \leq n|c_n|r^{n-1} \equiv M_n$. Since $\sum M_n < \infty$, the Weierstrass M test shows that $\sum h_n$ converges uniformly on $D(0,r)$. A uniform limit of continuous functions is continuous, hence $g = \sum h_n$ is continuous in $D(0,r)$ and in particular at z.

(iii) Part (ii) can be applied inductively. The equation $c_n = \frac{f^{(n)}(a)}{n!}$ can be found by differentiating the series n times.

(iv) If $f \equiv 0$ in some disc $D(a,\varepsilon)$, then $f^{(n)}(a) = 0$ for all n. Thus the power series is identically zero.

□

1.6 Exponentials

Definition. If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on $\mathbb{C}$, we say that f is *entire*.

Definition. The *complex exponential function* is defined by

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Proposition. (i) e^z is entire, and $(e^z)' = e^z$;
(ii) $e^z \neq 0$ and $e^{z+w} = e^z e^w$ for all complex z, w;
(iii) $e^{x+iy} = e^x(\cos y + i \sin y)$ for real x, y;
(iv) $e^z = 1$ if and only if $z = 2\pi n i$ for an integer n;
(v) if $z \in \mathbb{C}$, then there exists w such that $e^w = z$ if and only if $z \neq 0$.

Proof. (i) We can show that the radius of convergence is infinite. We can thus differentiate term by term and find $(e^z)' = e^z$.

(ii) Let $w \in \mathbb{C}$, and $F(z) = e^{z+w}e^{-w}$. Then we have

$$F'(z) = -e^{z+w}e^{-z} + e^{z+w}e^{-z} = 0$$

Hence $F(z)$ is constant. But $F(0) = e^w$, so $F(z) = e^w$. Taking $w = 0$, we have $e^z e^{-z} = 1$, so $e^z \neq 0$. Further, $e^{z+w} = e^z e^w$.

(iii) By part (ii), $e^{x+iy} = e^x e^{iy}$. Then, the series expansions of the sine and cosine functions can be used to finish the proof.

The rest of the proof is left as an exercise, which follows from (iii). □

1.7 Logarithms

Definition. Let $z \in \mathbb{C}$. Then, $w \in \mathbb{C}$ is a *logarithm* of z if $e^w = z$.

By part (v) above, z has a logarithm if and only if $z \neq 0$. In particular, $z \neq 0$ has infinitely many logarithms of the form $w + 2\pi i n$ for $n \in \mathbb{Z}$. If w is a logarithm of z, then $e^{\operatorname{Re} w} = |z|$, and hence $\operatorname{Re}(w) = \ln|z|$, where ln here is the unique real logarithm. In particular, $\operatorname{Re}(w)$ is uniquely determined by z.

Definition. Let $U \subset \mathbb{C} \setminus \{0\}$ be an open set. A *branch of logarithm* on U is a continuous function $\lambda : U \to \mathbb{C}$ such that $e^{\lambda(z)} = z$ for all $z \in U$.

Remark. Note that if λ is a branch of logarithm on U then λ is holomorphic in U with $\lambda'(z) = \frac{1}{z}$.

Proof. If $w \in U$ we have

$$\begin{aligned}
\lim_{z \to w} \frac{\lambda(z) - \lambda(w)}{z - w} &= \lim_{z \to w} \frac{\lambda(z) - \lambda(w)}{e^{\lambda(z)} - e^{\lambda(w)}} \\
&= \lim_{z \to w} \frac{1}{\left(\frac{e^{\lambda(z)} - e^{\lambda(w)}}{\lambda(z) - \lambda(w)} \right)} \\
&= \frac{1}{e^{\lambda(w)}} \lim_{z \to w} \frac{1}{\left(\frac{e^{\lambda(z) - \lambda(w)} - 1}{\lambda(z) - \lambda(w)} \right)} \\
&= \frac{1}{e^{\lambda(w)}} \lim_{h \to 0} \frac{1}{\left(\frac{e^{h} - 1}{h} \right)} \\
&= \frac{1}{e^{\lambda(w)}} \\
&= \frac{1}{w}
\end{aligned}$$

□

Definition. The *principal branch of logarithm* is the function

$$\operatorname{Log} : U_1 = \mathbb{C} \setminus \{x \in \mathbb{R} : x \leq 0\} \to \mathbb{C}; \quad \operatorname{Log}(z) = \ln|z| + i \arg(z)$$

where $\arg(z)$ is the unique argument of $z \in U_1$ in $(-\pi, \pi)$.

This is a branch of logarithm. Indeed, to check continuity, note that $z \mapsto \log|z|$ is continuous on $\mathbb{C} \setminus \{0\}$, and $z \mapsto \arg(z)$ is continuous since $\theta \mapsto e^{i\theta}$ is a homeomorphism $(-\pi, \pi) \to \mathbb{S}^1 \setminus \{-1\}$, and $z \mapsto \frac{z}{|z|}$ is continuous on $\mathbb{C} \setminus \{0\}$. Further,

$$e^{\operatorname{Log}(z)} = e^{\ln|z|} e^{i \arg(z)} = |z|(\cos \arg z + i \sin \arg z) = z$$

Note that Log cannot be continuously extended to $\mathbb{C} \setminus \{0\}$, since $\arg z \to \pi$ as $z \to -1$ with $\operatorname{Im}(z) > 0$, and $\arg z \to -\pi$ as $z \to -1$ with $\operatorname{Im}(z) < 0$. We will later prove that no branch of logarithm can exist on all of $\mathbb{C} \setminus \{0\}$.

Proposition. (i) Log is holomorphic on U_1 with $(\operatorname{Log} z)' = \frac{1}{z}$; and

(ii) for $|z| < 1$, we have

$$\operatorname{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}$$

Proof. Part (i) follows from the above. The radius of convergence of the given series is one, and $1 + z \in U_1$, so both sides of the equation are defined on the unit disc. Then,

$$F(z) = \operatorname{Log}(1+z) - \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n} \implies F'(z) = \frac{1}{1+z} - \sum_{n=1}^{\infty} (-z)^{n-1} = 0 \implies F(z) = F(0) = 0$$

□

We can now define the *principal branch of* z^α by

$$z^\alpha = e^{\alpha \operatorname{Log}(z)}$$

Note that z^α is holomorphic on U_1 with $(z^\alpha)' = \alpha z^{\alpha-1}$. We can use exponentials to define the trigonometric and hyperbolic functions, which are all entire functions with derivatives matching those of the real definitions of these functions.

1.8 Conformality

Let $f : U \to \mathbb{C}$ be holomorphic, where U is an open set. Let $w \in U$ and suppose that $f'(w) \neq 0$. Let $\gamma_1, \gamma_2 : [-1, 1] \to U$ be C^1 curves, such that $\gamma_i(0) = w$ and $\gamma_i'(0) \neq 0$. Then $f \circ \gamma_i$ are C^1 curves passing through $f(w)$. Further, $(f \circ \gamma_i)'(0) = f'(w)\gamma_i'(0) \neq 0$. Thus

$$\frac{(f \circ \gamma_1)'(0)}{(f \circ \gamma_2)'(0)} = \frac{\gamma_1'(0)}{\gamma_2'(0)}$$

Hence,

$$\arg(f \circ \gamma_1)'(0) - \arg(f \circ \gamma_2)'(0) = \arg \gamma_1'(0) - \arg \gamma_2'(0)$$

In other words, the angle that the curves make when they intersect at w is the same angle that their images $f \circ \gamma_i$ make when they intersect at $f(w)$, and the orientation also is preserved (clockwise or anticlockwise). Hence, f is angle-preserving at w whenever $f'(w) \neq 0$. In particular, if γ_i are tangential at w, the curves $f \circ \gamma_i$ are tangential at $f(w)$.

Remark. If f is C^1, then the converse holds. If $w \in U$ and $(f \circ \gamma)'(0) \neq 0$ for any C^1 curve γ with $\gamma(0) = w$ and $\gamma'(0) \neq 0$, and if f is angle-preserving at w in the above sense, then $f'(w)$ exists and is nonzero.

Definition. A holomorphic function $f : U \to \mathbb{C}$ on an open set U is *conformal* at $w \in U$ if $f'(w) \neq 0$.

Definition. Let $U, \widetilde{U}$ be domains in $\mathbb{C}$. A map $f : U \to \widetilde{U}$ is a *conformal equivalence* between $U, \widetilde{U}$ if f is a bijective holomorphic map with $f'(z) \neq 0$ for all $z \in U$.

Remark. We will prove later that if f is holomorphic and injective, then $f'(z) \neq 0$ for all z. Thus, in the above definition, the condition $f'(z) \neq 0$ is redundant.

Remark. It is automatic that $f^{-1}: \widetilde{U} \to U$ is holomorphic, which will follow from the holomorphic inverse function theorem.

Example. Möbius maps

$$f(z) = \frac{az+b}{cz+d}$$

are conformal on $\mathbb{C} \setminus \{-d/c\}$ if $c \neq 0$, and conformal on $\mathbb{C}$ if $c = 0$. Möbius maps are sometimes used as explicit conformal equivalences between subdomains of $\mathbb{C}$. For instance, let $\mathbb{H}$ be the open upper half plane in $\mathbb{C}$. Then

$$z \in \mathbb{H} \iff |z-i| < |z+i| \iff \left|\frac{z-i}{z+i}\right| < 1$$

Thus the map $z \mapsto \frac{z-i}{z+i}$ maps $\mathbb{H}$ onto $D(0,1)$, so g is a conformal equivalence.

Example. Let $f: z \mapsto z^n$ for $n \geq 1$. Then

$$f: \left\{z \in \mathbb{C} \setminus \{0\}: 0 < \arg z < \frac{\pi}{n}\right\} \to \mathbb{H}$$

is the restricted map on a sector. The restricted f is a conformal equivalence with $f^{-1}(z) = z^{1/n}$, the principal branch of $z^{1/n}$.

Example. The function

$$\exp: \{z \in \mathbb{C}: -\pi < \operatorname{Im} z < \pi\} \to \mathbb{C} \setminus \{x \in \mathbb{R}: x \leq 0\}$$

is a conformal equivalence, with inverse Log.

Theorem (Riemann mapping theorem)**.** *This theorem is non-examinable.*
Any simply connected domain $U \subset \mathbb{C}$ with $U \neq \mathbb{C}$ is conformally equivalent to $D(0,1)$.

2 Integration

2.1 Introduction

Definition. If $f: [a,b] \subset \mathbb{R} \to \mathbb{C}$ is a complex function, and the real and imaginary parts of f are Riemann integrable, then we define

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b \operatorname{Re}(f(t))\,\mathrm{d}t + i\int_a^b \operatorname{Im}(f(t))\,\mathrm{d}t$$

In particular, for $g: [a,b] \to \mathbb{R}$, we have

$$\int_a^b ig(t)\,\mathrm{d}t = i\int_a^b g(t)\,\mathrm{d}t$$

Thus, for a complex constant $w \in \mathbb{C}$, we can find

$$\int_a^b w f(t)\,\mathrm{d}t = w \int_a^b f(t)\,\mathrm{d}t$$

Proposition (basic estimate). If $f : [a, b] \to \mathbb{C}$ is continuous, then

$$\left| \int_a^b f(t)\,\mathrm{d}t \right| \leq \int_a^b |f(t)|\,\mathrm{d}t \leq (b-a) \sup_{t \in [a,b]} |f(t)|$$

Equality holds if and only if f is constant.

Proof. If $\int_a^b f(t)\,\mathrm{d}t = 0$ then the proof is complete. Otherwise, we can write the value of the integral as $re^{i\theta}$ for $\theta \in [0, 2\pi)$. Let $M = \sup_{t \in [a,b]} |f(t)|$. Then we have

$$\begin{aligned}
\left| \int_a^b f(t)\,\mathrm{d}t \right| &= r \\
&= e^{-i\theta} \int_a^b f(t)\,\mathrm{d}t \\
&= \int_a^b e^{-i\theta} f(t)\,\mathrm{d}t \\
&= \int_a^b \mathrm{Re}(e^{-i\theta} f(t))\,\mathrm{d}t + i \int_a^b \mathrm{Im}(e^{-i\theta} f(t))\,\mathrm{d}t
\end{aligned}$$

Since the left hand side is real, the imaginary integral vanishes.

$$\begin{aligned}
\left| \int_a^b f(t)\,\mathrm{d}t \right| &= \int_a^b \mathrm{Re}(e^{-i\theta} f(t))\,\mathrm{d}t \\
&\leq \int_a^b \left| e^{-i\theta} f(t) \right| \mathrm{d}t = \int_a^b |f(t)|\,\mathrm{d}t \\
&\leq (b-a)M
\end{aligned}$$

Equality holds if and only if $|f(t)| = M$ and $\mathrm{Re}(e^{-i\theta} f(t)) = M$ for all $t \in [a, b]$, which is true only if $|f(t)| = M$ and $\arg(f(t)) = \theta$ hence $f = Me^{i\theta}$ for all t. □

2.2 Integrating along curves

Definition. Let $U \subset \mathbb{C}$ be an open set and let $f : U \to \mathbb{C}$ be continuous. Let $\gamma : [a, b] \to U$ be a C^1 curve. Then the *integral of f along γ* is

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

This definition is consistent with the previous definition of the integral of a function f along the interval $[a, b]$. The integral along a curve has various convenient properties.

(i) It is invariant under the choice of parametrisation. Let $\varphi\colon [a_1, b_1] \to [a, b]$ be C^1 and injective with $\varphi(a_1) = a$ and $\varphi(b_1) = b$. Let $\delta = \gamma \circ \varphi\colon [a_1, b_1] \to U$. Then

$$\int_\delta f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z$$

Indeed,

$$\begin{aligned}\int_\delta f(z)\,\mathrm{d}z &= \int_{a_1}^{b_1} f(\gamma(\varphi(t)))\gamma'(\varphi(t))\varphi'(t)\,\mathrm{d}t \\ &= \int_a^b f(\gamma(s))\gamma'(s)\,\mathrm{d}s \\ &= \int_\gamma f(z)\,\mathrm{d}z\end{aligned}$$

(ii) The integral is linear. It is easy to check that

$$\int_\gamma (\lambda f(z) + \mu g(z))\,\mathrm{d}z = \lambda \int_\gamma f(z)\,\mathrm{d}z + \mu \int_\gamma g(z)\,\mathrm{d}z$$

for complex constants $\lambda, \mu \in \mathbb{C}$.

(iii) The additivity property states that if $\gamma\colon [a, b] \to U$ is C^1 and $a < c < b$, then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma|_{[a,c]}} f(z)\,\mathrm{d}z + \int_{\gamma|_{c,b}} f(z)\,\mathrm{d}z$$

(iv) We define the *inverse path* $(-\gamma)\colon [-b, -a] \to U$ by $(-\gamma)(t) = \gamma(-t)$. Then

$$\int_{(-\gamma)} f(z)\,\mathrm{d}z = -\int_\gamma f(z)\,\mathrm{d}z$$

Definition. Let $\gamma\colon [a, b] \to \mathbb{C}$ be a C^1 curve. Then the *length* of γ is

$$\mathrm{length}(\gamma) = \int_a^b |\gamma'(t)|\,\mathrm{d}t$$

Definition. A *piecewise C^1 curve* is a continuous map $\gamma\colon [a, b] \to \mathbb{C}$ such that there exists a finite subdivision

$$a = a_0 < a_1 < \cdots < a_n = b$$

such that each $\gamma_j = \gamma|_{[a_{j-1}, a_j]}$ is C^1 for $1 \le j \le n$. Then, for such a piecewise C^1 curve, we define

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^n \int_{\gamma_j} f(z)\,\mathrm{d}z$$

and

$$\operatorname{length}(\gamma) = \sum_{j=1}^n \operatorname{length}(\gamma_j) = \sum_{j=1}^n \int_{a_{j-1}}^{a_j} |\gamma'(t)|\,\mathrm{d}t$$

By the additivity property, both definitions are invariant under changing the subdivision. From here, we will use 'curve' to refer to 'piecewise C^1 curve', unless stated otherwise.

Definition. If $\gamma_1 \colon [a,b] \to \mathbb{C}$ and $\gamma_2 \colon [c,d]$ are curves with $\gamma_1(b) = \gamma_2(c)$, we define the *sum* of γ_1 and γ_2 to be the curve

$$(\gamma_1 + \gamma_2) \colon [a, b+d-c] \to \mathbb{C}; \quad (\gamma_1+\gamma_2)(t) = \begin{cases} \gamma_1(t) & a \leq t \leq b \\ \gamma_2(t-b+c) & b \leq t \leq b+d-c \end{cases}$$

Proposition. Let $f \colon U \to \mathbb{C}$ be continuous and $\gamma \colon [a,b] \to \mathbb{C}$, we have

$$\left| \int_\gamma f(z)\,\mathrm{d}z \right| \leq \operatorname{length}(\gamma) \sup_\gamma |f|$$

where $\sup_\gamma g \equiv \sup_{t \in [a,b]} g(\gamma(t))$.

Proof. If γ is C^1, then

$$\left| \int_\gamma f(z)\,\mathrm{d}z \right| = \left| \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t \right| \leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\,\mathrm{d}t \leq \sup_{t\in[a,b]} |f(\gamma(t))| \operatorname{length}(\gamma)$$

If γ is piecewise C^1, then the result follows from the definition of a piecewise C^1 function and the above. □

2.3 Fundamental theorem of calculus

Theorem (fundamental theorem of calculus)**.** Let $f \colon U \to \mathbb{C}$ be continuous on an open set $U \subset \mathbb{C}$. Let $F \colon U \to \mathbb{C}$ be a function such that $F'(z) = f(z)$ for all $z \in U$. Then, for any curve $\gamma \colon [a,b] \to U$, we have

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$

If γ is a closed curve, then $\int_\gamma f(z) = 0$. Such a function F is known as an *antiderivative* of f.

Proof.

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}F(\gamma(t))\,\mathrm{d}t = F(\gamma(b)) - F(\gamma(a))$$

□

Remark. Note that we assume that F exists such that $F'(z) = f(z)$; such an F is not provided for by the theorem.

Example. For an integer n and the curve $\gamma(t) = Re^{2\pi it}$ for $t = [0,1]$, consider the integral $\int_\gamma z^n\,\mathrm{d}z$. For $n \neq -1$, the function $\frac{z^{n+1}}{n+1}$ is an antiderivative of z^n. Hence, $\int_\gamma z^n\,\mathrm{d}z = 0$ since γ is a closed curve. If $n = -1$, we can use the definition of the integral to find

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = \int_0^1 \frac{1}{\gamma(t)}\gamma'(t)\,\mathrm{d}t = \int_0^1 \frac{1}{Re^{2\pi it}}2\pi iRe^{2\pi it}\,\mathrm{d}t = 2\pi i$$

This is not zero, hence for all $R > 0$, $\frac{1}{z}$ has no antiderivative in any open set containing the circle $\{|z| = R\}$. In particular, for any branch of logarithm λ, it has derivative $\frac{1}{z}$, hence there exists no branch of logarithm on $\mathbb{C}^\star = \mathbb{C} \setminus \{0\}$.

Theorem (converse to fundamental theorem of calculus)**.** Let $U \subset \mathbb{C}$ be a domain. If $f : U \to \mathbb{C}$ is continuous and if $\int_\gamma f(z)\,\mathrm{d}z = 0$ for every closed curve γ in U, then f has an antiderivative. In other words, there exists a holomorphic function $F : U \to \mathbb{C}$ such that $F' = f$ in U.

Proof. Let $a_0 \in U$. Then for $w \in U$, we can define

$$F(w) = \int_{\gamma_w} f(z)\,\mathrm{d}z$$

where $\gamma_w : [0,1] \to \mathbb{C}$ is a curve with $\gamma_w(0) = a_0$ and $\gamma_w(1) = w$.

The definition of F is independent of the choice of γ_w. Indeed, suppose two paths γ_w, γ_w' exist. Then the curve $\gamma_w + (-\gamma_w')$ is a closed path, and by assumption the integral along this curve is zero. Thus F is independent of the choice of path as claimed. So F is a well-defined function.

Now, let $w \in U$. Since U is an open set, there exists $r > 0$ such that $D(w,r) \subset U$. For $h \in \mathbb{C}$ with $0 < |h| < r$, let δ_h be the radial path $t \mapsto w + th$ for $t \in [0,1]$. Now we define

$$\gamma = \gamma_w + \delta_h + (-\gamma_{w+h})$$

This is a closed curve contained within U, hence $\int_\gamma f(z)\,\mathrm{d}z = 0$. Thus

$$\int_{\gamma_{w+h}} f(z)\,\mathrm{d}z = \int_{\gamma_w} f(z)\,\mathrm{d}z + \int_{\delta_h} f(z)\,\mathrm{d}z$$

Informally, the integral has an additivity property which is independent of the path taken. Rewriting this using F,

$$\begin{aligned} F(w+h) &= F(w) + \int_{\delta_h} f(z)\,\mathrm{d}z \\ &= F(w) + \int_{\delta_h} (f(z) + f(w) - f(w))\,\mathrm{d}z \\ &= F(w) + hf(w) + \int_{\delta_h} (f(z) - f(w))\,\mathrm{d}z \end{aligned}$$

Hence, by continuity of f,

$$\begin{aligned} \left|\frac{F(w+h)-F(w)}{h} - f(w)\right| &= \frac{1}{|h|}\left|\int_{\delta_h} (f(z)-f(w))\,\mathrm{d}z\right| \\ &\leq \frac{1}{|h|}\mathrm{length}(\delta_h) \sup_{z \in \mathrm{Im}\,\delta_h} |f(z)-f(w)| \\ &= \sup_{z \in \mathrm{Im}\,\delta_h} |f(z)-f(w)| \\ \therefore \lim_{h\to 0}\left|\frac{F(w+h)-F(w)}{h} - f(w)\right| &= \lim_{h \to 0} \sup_{z \in \mathrm{Im}\,\delta_h} |f(z)-f(w)| = 0 \end{aligned}$$

Thus, F is differentiable at w with $F'(w) = f(w)$. □

2.4 Star-shaped domains

Definition. A domain U is *star-shaped*, or a *star domain*, if there exists a (not necessarily unique) centre $a_0 \in U$ such that for all $w \in U$, the straight line segment $[a_0, w]$ is contained within U.

Remark. Any disc is convex; any convex domain is star-shaped; any star-shaped domain is path-connected. The reverse implications are not true in general.

Definition. A *triangle* in $\mathbb{C}$ is the *convex hull* of three points in $\mathbb{C}$. The (closed) convex hull of a set S is the smallest (closed) convex set C such that $S \subseteq C$. In this case, if $z_1, z_2, z_3 \in \mathbb{C}$, we have

$$T = \{az_1 + bz_2 + cz_3 : 0 \leq a, b, c \leq 1, a + b + c = 1\}$$

When used as a curve, the boundary ∂T represents the piecewise affine closed curve $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ where γ_i are affine functions parametrising the three line segments on the boundary of T.

Corollary. Let U be a star-shaped domain. Let $f : U \to \mathbb{C}$ be continuous and $\int_{\partial T} f(z)\,\mathrm{d}z = 0$ for any triangle $T \subset U$. Then f has an antiderivative in U.

Remark. This is a relaxation of the conditions from the previous theorem.

Proof. Let a_0 be a centre for the domain U. Let w be an arbitrary point in U. Then let γ_w be the affine function parametrising the directed line segment $[a_0, w]$, and let $F(w) = \int_{\gamma_w} f(z)\,\mathrm{d}z$. Using h and δ_h as above, by letting $\gamma = \gamma_w + \delta_h + (-\gamma_{w+h})$ we then have $\int f(z)\,\mathrm{d}z = \pm \int_{\partial T} f(z)\,\mathrm{d}z$ for a triangle $T \subset U$. Since the integral around a triangle is zero by hypothesis, $\int_\gamma f(z)\,\mathrm{d}z = 0$. We then complete the proof in analogous way to the general case. □

Theorem (Cauchy's theorem for triangles)**.** Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ be a holomorphic function. Then $\int_{\partial T} f(z)\,\mathrm{d}z = 0$ for all triangles $T \subset U$.

Proof. Let $\eta(t) = \int_{\partial T} f(z)\,\mathrm{d}z$. We will subdivide the triangle T into four smaller triangles $T^{(1)}, T^{(2)}, T^{(3)}, T^{(4)}$. The vertices of the inner triangle are the midpoints of the sides of T, and the three other triangles are constructed to fill the remaining area of T. Thus,

$$\eta(T) = \int_{\partial T^{(1)}} f(z)\,\mathrm{d}z + \int_{\partial T^{(2)}} f(z)\,\mathrm{d}z + \int_{\partial T^{(3)}} f(z)\,\mathrm{d}z + \int_{\partial T^{(4)}} f(z)\,\mathrm{d}z$$

Then, by the triangle inequality, there exists a triangle $T^{(j)}$ such that

$$\left|\int_{\partial T^{(j)}} f(z)\,\mathrm{d}z\right| \geq \frac{|\eta(T)|}{4}$$

Let $T_0 = T$, and $T_1 = T^{(j)}$, so $|\eta(T_1)| \geq \frac{1}{4}|\eta(T_0)|$. We can show geometrically that for any choice of T_i, $\text{length}(\partial T_1) = \frac{1}{2}\text{length}(\partial T_0)$. Inductively, we can subdivide T_i and produce T_{i+1}, such that

$$T_0 \supset T_1 \supset \cdots; \quad |\eta(T_n)| \geq \frac{1}{4}|\eta(T_{n-1})|; \quad \text{length}(\partial T_n) = \frac{1}{2}\text{length}(\partial T_{n-1})$$

Hence,

$$|\eta(T_n)| \geq \frac{1}{4^n}|\eta(T_0)|; \quad \text{length}(\partial T_n) = \frac{1}{2^n}\text{length}(\partial T_0)$$

Since T_n are non-empty, nested closed subsets with diameter converging to zero, we can show that $\bigcap_{n=1}^\infty T_n = \{z_0\}$ for some $z_0 \in \mathbb{C}$. Let $\varepsilon > 0$. Since f is differentiable at z_0, there exists $\delta > 0$ such that

$$\begin{aligned} z \in U, |z - z_0| < \delta \implies & \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| \leq \varepsilon \\ \implies & |f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \varepsilon|z - z_0| \end{aligned}$$

Now, observe that for all n,

$$\int_{\partial T_n} f(z)\,\mathrm{d}z = \int_{\partial T_n} (f(z) - f(z_0) - f'(z_0)(z - z_0))\,\mathrm{d}z$$

since $\int_{\partial T_n} \mathrm{d}z = \int_{\partial T_n} z\,\mathrm{d}z = 0$ by the fundamental theorem of calculus. Let n such that $T_n \subset D(z_0, \delta)$.

Hence,

$$
\begin{aligned}
\frac{1}{4^n}|\eta(T_0)| &\leq |\eta(T_n)| \\
&= \left|\int_{\partial T_n} f(z)\,\mathrm{d}z\right| \\
&= \left|\int_{\partial T_n} (f(z) - f(z_0) - f'(z_0)(z - z_0))\,\mathrm{d}z\right| \\
&\leq \left(\sup_{z\in\partial T_n} |f(z) - f(z_0) - f'(z_0)(z - z_0)|\right)\mathrm{length}(\partial T_n) \\
&\leq \varepsilon\left(\sup_{z\in\partial T_n} |z - z_0|\right)\mathrm{length}(\partial T_n) \\
&\leq \varepsilon \cdot \mathrm{length}(\partial T_n)^2 \\
&= \frac{\varepsilon}{4^n}\mathrm{length}(\partial T_0)^2 \\
\therefore |\eta(T_0)| &\leq \varepsilon \cdot \mathrm{length}(\partial T_0)^2
\end{aligned}
$$

ε was arbitrary, hence $\eta(T_0)$ must be zero. □

We can generalise the above theorem for functions that are holomorphic except at a finite number of points.

Theorem. Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ be a continuous function. Let $S \subset U$ be a finite set and suppose that f is holomorphic on $U \setminus S$. Then $\int_{\partial T} f(z)\,\mathrm{d}z = 0$ for all triangles $T \subset U$.

Proof. By the procedure above, we can subdivide T into a total of 4^n smaller triangles; at each step we join the midpoints of the sides of the triangles of the previous step. We will keep all of the smaller triangles, and let the sequence of such smaller triangles be denoted $T_1, \dots, T_N$. Then, since the integrals along the sides of the smaller triangles that are interior to T cancel, we have

$$\int_{\partial T} f(z)\,\mathrm{d}z = \sum_{j=1}^{N} \int_{\partial T_j} f(z)\,\mathrm{d}z$$

By the previous theorem, $\int_{\partial T_j} f(z)\,\mathrm{d}z = 0$ unless T_j intersects with S. So by letting $I = \{j : T_j \cap S \neq \varnothing\}$, we have

$$\int_{\partial T} f(z)\,\mathrm{d}z = \sum_{j\in I} \int_{\partial T_j} f(z)\,\mathrm{d}z$$

Since any point may be in at most six of the smaller triangles, and $\mathrm{length}(\partial T_j) = \frac{1}{2^n}\mathrm{length}(\partial T)$, we find

$$\left|\int_{\partial T} f(z)\,\mathrm{d}z\right| \leq 6|S|\left(\sup_{z\in T} |f(z)|\right)\frac{\mathrm{length}(\partial T)}{2^n}$$

Then let $n \to \infty$ and the result then holds as required. □

We can now prove the 'convex Cauchy' theorem.

Corollary (Cauchy's theorem for convex sets)**.** Let $U \subset \mathbb{C}$ be convex, or more generally, a star domain. Let $f : U \to \mathbb{C}$ be continuous on U and holomorphic in $U \setminus S$ where S is a finite set. Then $\int_\gamma f(z)\,\mathrm{d}z = 0$ for any closed curve γ in U.

Proof. By the theorems above, $\int_{\partial T} f(z)\,\mathrm{d}z = 0$ for any triangle $T \subset U$. Since U is a star domain and f is continuous, this means that f has an antiderivative in U. The result then follows from the fundamental theorem of calculus. □

Remark. We will soon show that if $f : U \to \mathbb{C}$ is continuous and holomorphic in $U \setminus S$ where S is finite, then f is holomorphic in U.

2.5 Cauchy's integral formula

For a disc $D(a,\rho)$ we will write $\int_{\partial D(a,\rho)} f(z)\,\mathrm{d}z$ to mean $\int_\gamma f(z)\,\mathrm{d}z$ where $\gamma : [0,1] \to \mathbb{C}$ is the curve $\gamma(t) = a + \rho e^{2\pi i t}$.

Theorem (Cauchy's integral formula for a disc)**.** Let $D = D(a,r)$ and let $f : D \to \mathbb{C}$ be holomorphic. Then, for any ρ with $0 < \rho < r$ and any $w \in D(a,\rho)$, we have

$$f(w) = \frac{1}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(z)\,\mathrm{d}z}{z-w}$$

In particular, taking $w = a$,

$$f(a) = \frac{1}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(z)\,\mathrm{d}z}{z-a} = \int_0^1 f(a + \rho e^{2\pi i t})\,\mathrm{d}t$$

This final equation is called the *mean value property* for holomorphic functions.

We first need the following lemma.

Lemma. If $\gamma : [a,b] \to \mathbb{C}$ is a curve and (f_n) is a sequence of continuous complex functions on $\operatorname{Im}\gamma$ converging uniformly to f on $\operatorname{Im}\gamma$, then $\int_\gamma f_n(z)\,\mathrm{d}z \to \int_\gamma f(z)\,\mathrm{d}z$.

Proof. We have

$$\left|\int_\gamma f_n(z)\,\mathrm{d}z - \int_\gamma f(z)\,\mathrm{d}z\right| = \left|\int_\gamma (f_n(z) - f(z))\,\mathrm{d}z\right| \le \sup_{z \in \operatorname{Im}\gamma} |f_n(z) - f(z)| \operatorname{length}(\gamma)$$

□

We can now prove Cauchy's integral formula for a disc.

Proof. Let $w \in D(a,\rho)$ be fixed, and define $h : D \to \mathbb{C}$ by

$$h(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w \end{cases}$$

Then h is continuous on D and holomorphic in $D \setminus \{w\}$. By Cauchy's theorem for convex sets,

$$\int_{\partial D(a,\rho)} h(z)\,\mathrm{d}z = 0$$

Substituting for h, we find

$$f(w)\int_{\partial D(a,\rho)} \frac{\mathrm{d}z}{z-w} = \int_{\partial D(a,\rho)} \frac{f(z)\,\mathrm{d}z}{z-w}$$

It now suffices to prove that

$$\int_{\partial D(a,\rho)} \frac{\mathrm{d}z}{z-w} = 2\pi i$$

Note that

$$\frac{1}{z-w} = \frac{1}{z-a+a-w} = \frac{1}{(z-a)\left(1-\frac{w-a}{z-a}\right)} = \sum_{j=0}^{\infty} \frac{(w-a)^j}{(z-a)^{j+1}}$$

where the convergence is uniform for $z \in \partial D(a,\rho)$ by the Weierstrass M-test. Therefore, by the above lemma, we interchange summation and integration to find

$$\int_{\partial D(a,\rho)} \frac{\mathrm{d}z}{z-w} = \sum_{j=0}^{\infty}(w-a)^j \int_{\partial D(a,\rho)} \frac{\mathrm{d}z}{(z-a)^{j+1}}$$

For $j \geq 1$, the function $\frac{1}{(z-a)^{j+1}}$ has an antiderivative in a neighbourhood of $\partial D(a,\rho)$, hence all integrals on the right hand side for $j \geq 1$ vanish. For $j = 0$, we can compute directly that $\int_{\partial D(a,\rho)} \frac{\mathrm{d}z}{z-a} = 2\pi i$. Hence, $\int_{\partial D(a,\rho)} \frac{\mathrm{d}z}{z-w} = 2\pi i$, proving Cauchy's integral formula.

Now, taking $w = a$ in Cauchy's integral formula, we find

$$f(a) = \frac{1}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(z)\,\mathrm{d}z}{z-a}$$

By direct computation using the parametrisation $t \mapsto a + \rho e^{2\pi i t}$ for $t \in [0,1]$, we find

$$f(a) = \int_0^1 f(a+\rho e^{2\pi i t})\,\mathrm{d}t$$

as required. □

2.6 Liouville's theorem

Theorem. Let $f : \mathbb{C} \to \mathbb{C}$ be entire and bounded. Then f is constant. More generally, if f is entire with sublinear growth (there exist $K \geq 0$ and $\alpha < 1$ such that $|f(z)| \leq K(1+|z|^{\alpha})$ for all $z \in \mathbb{C}$) then f is constant.

Proof. Let $w \in \mathbb{C}$ and $\rho > |w|$. By Cauchy's integral formula, we have

$$f(w) = \frac{1}{2\pi i} \int_{\partial D(0,\rho)} \frac{f(z)\,\mathrm{d}z}{z - w}; \quad f(0) = \frac{1}{2\pi i} \int_{\partial D(0,\rho)} \frac{f(z)\,\mathrm{d}z}{z}$$

Thus,

$$\begin{aligned} |f(w) - f(0)| &= \frac{1}{2\pi} \left| \int_{\partial D(0,\rho)} \frac{w f(z)\,\mathrm{d}z}{z(z-w)} \right| \\ &\leq \frac{|w|}{2\pi} \sup_{z \in \partial D(0,\rho)} \frac{|f(z)|}{|z| \cdot ||z| - |w||} \operatorname{length}(\partial D(0,\rho)) \\ &\leq \frac{|w| K (1 + \rho^\alpha)}{2\pi\rho(\rho - |w|)} 2\pi\rho \\ &= \frac{|w| K (1 + \rho^\alpha)}{\rho - |w|} \end{aligned}$$

By letting $\rho \to \infty$, we can conclude $f(w) = f(0)$. □

Theorem (fundamental theorem of algebra)**.** Every non-constant polynomial with complex coefficients has a complex root.

Proof. Let $p(z) = a_n z^n + \dots + a_0$ be a complex polynomial of degree $n \geq 1$. Then $a_n \neq 0$, and for $z \neq 0$ we can write

$$p(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n} \right)$$

By the triangle inequality,

$$|p(z)| \geq |z|^n \left(|a_n| - \frac{|a_{n-1}|}{|z|} - \dots - \frac{|a_0|}{|z|^n} \right)$$

Hence, there exists $R > 0$ such that $|p(z)| \geq 1$ for $|z| > R$.

Now, if $p(z) \neq 0$ for all z, then $g(z) = \frac{1}{p(z)}$ is entire. By the above, $|g(z)| \leq 1$ for $|z| > R$. By continuity of g, we have further that $|g(z)|$ is bounded from above on the compact set $\{|z| \leq R\}$. Hence, g is a bounded entire function. By Liouville's theorem, g is constant. Since p is non-constant, this is a contradiction. Hence p has a zero. □

Theorem (local maximum modulus principle)**.** Let $f : D(a, R) \to \mathbb{C}$ be holomorphic, and $|f(z)| \leq |f(a)|$ for all $z \in D(a, R)$. Then f is constant.

Proof. By the mean value property,

$$f(a) = \int_0^1 f(a + \rho e^{2\pi i t})\,\mathrm{d}t$$

Therefore,

$$|f(a)| = \left| \int_0^1 f(a + \rho e^{2\pi i t})\,\mathrm{d}t \right| \leq \sup_{t \in [0,1]} |f(a + \rho e^{2\pi i t})| \leq |f(a)|$$

where the last inequality is by hypothesis. Therefore, both inequalities must be equalities. Equality in the first inequality implies that $f(a+\rho e^{2\pi i t}) = c_\rho$ for some constant c_ρ and all $t \in [0,1]$. Then, by the first equality, $|c_\rho| = |f(a)|$ for all $\rho \in (0,R)$. Thus, $|f(a+\rho e^{2\pi i t})|$ is constant for all $\rho \in (0,R)$ and $t \in [0,1]$. Hence $|f(z)|$ is constant on $D(a,R)$. By the Cauchy–Riemann equations, f must be constant. □

2.7 Taylor series

Theorem. Let $f : D(a,R) \to \mathbb{C}$ be holomorphic. Then f has a convergent power series representation on $D(a,R)$. More precisely, there exists a sequence of complex numbers $c_0, c_1, \ldots$ such that

$$f(w) = \sum_{n=0}^{\infty} c_n (w-a)^n$$

The coefficient c_n is given by

$$c_n = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)\,\mathrm{d}z}{(z-a)^{n+1}}$$

for any $\rho \in (0,R)$.

Proof. Let $0 < \rho < R$. Then, for any $w \in D(0,\rho)$, we have by Cauchy's integral formula that

$$\begin{aligned} f(w) &= \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)\,\mathrm{d}z}{z-w} \\ &= \frac{1}{2\pi i} \int_{\partial D(a,\rho)} f(z) \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}}\,\mathrm{d}z \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)\,\mathrm{d}z}{(z-a)^{n+1}} \right) (w-a)^n \end{aligned}$$

The last equality holds since the series under the integral converges uniformly for all $z \in \partial D(a,\rho)$. Let

$$c_n(\rho) = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)\,\mathrm{d}z}{(z-a)^{n+1}}$$

Then we have shown that $f(w) = \sum_{n=0}^{\infty} c_n(\rho)(w-a)^{n+1}$ for all $w \in D(a,\rho)$. By a previous theorem, the function f has derivatives of all orders in $D(a,\rho)$ and hence $c_n(\rho) = \frac{f^{(n)}(a)}{n!}$, which is independent of ρ, so we can let $c_n = c_n(\rho)$ for an arbitrary ρ. □

Corollary. If f is holomorphic on an open set $U \subset \mathbb{C}$, then f has derivatives of all orders in U, and those derivatives are holomorphic on U.

Proof. We have a power series representation for f near every points, so its derivatives of all orders exist everywhere. Hence, the derivatives of all orders are holomorphic. □

Remark. We can explicitly compute from the c_n above that

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)\,\mathrm{d}z}{(z-a)^{n+1}}$$

This is a special case of a Cauchy integral formula for derivatives.

Note also that by taking $n = 1$, we can apply the estimate for the integral to find

$$|f'(a)| \leq \frac{1}{\rho}\left(\sup_{z \in \partial D(a,\rho)} |f(z)| \right)$$

This can be thought of as a localised version of Liouville's theorem, and it directly implies Liouville's theorem. Indeed, if f is entire and bounded, let $a \in \mathbb{C}$ and by applying the estimate and letting $\rho \to \infty$ we can conclude $f' = 0$ on $\mathbb{C}$, giving that f is constant.

Definition. A function f is *analytic* at a point $a \in \mathbb{C}$ (or $\mathbb{R}$) if there exists a neighbourhood of a such that f is given by a convergent power series about a.

Remark. If f is analytic at a, we must have derivatives of all orders of f near a. The above corollary implies that if f is complex, the following are equivalent.

(i) f is analytic at a

(ii) f has complex derivatives of all orders in a neighbourhood of a

(iii) f is complex differentiable once in a neighbourhood in a neighbourhood of a (so f is holomorphic at a)

For real functions, this is not the case. For example, consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \exp(-x^{-2})$. This has $f^{(n)}(0) = 0$ for all n, so f is not given by a convergent power series near zero.

Let $U \subset \mathbb{C}$ be an open set. Now, we have that $f = u + iv$ is holomorphic in U if and only if u and v have continuous partial derivatives in U, and that $u.v$ satisfy the Cauchy–Riemann equations. Further, the corollary above implies that u, v are C^2 functions. This shows that u and v are harmonic.

Theorem (Morera's theorem)**.** Let $U \subseteq \mathbb{C}$ be open, and $f : U \to \mathbb{C}$ be a continuous function such that $\int_\gamma f(z)\,\mathrm{d}z = 0$ for all closed curves γ in U. Then f is holomorphic in U.

Remark. This can be thought of as a converse to Cauchy's theorem.

Proof. We know that f has a holomorphic antiderivative F on U. Then, we know that F is twice differentiable in U. Since $F' = f$, f is holomorphic. □

Corollary. Let $U \subseteq \mathbb{C}$ be an open set. Let $f : U \to \mathbb{C}$ be a continuous function and holomorphic in $U \setminus S$, where S is a finite set. Then f is holomorphic in U.

Proof. For all $a \in U$, there exists $r > 0$ such that $D = D(a, r) \subset U$. Since D is convex, we can apply Cauchy's formula for convex sets to observe that $\int_\gamma f(z)\,\mathrm{d}z = 0$ for all closed curves in D. Then by Morera's theorem, f is holomorphic. □

2.8 Zeroes of holomorphic functions

Definition. Let f be a holomorphic function on a disc $D = D(a, R)$. By the Taylor series theorem, there exist constants c_n such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

for all $z \in D$. Then if f is not identically zero, there exists n such that $c_n \neq 0$. Let $m = \min\{n : c_n \neq 0\}$. Then,

$$f(z) = (z-a)^m g(z); \quad g(z) = \sum_{n=m}^{\infty} c_n (z-a)^{n-m}$$

Note that g is holomorphic on D, and $g(a) = c_m \neq 0$.
If $m \neq 0$, we say that f has a *zero of order m at $z = a$*. Hence m is the smallest natural number n such that $f^{(n)}(a) \neq 0$. If $S \subseteq \mathbb{C}$, then a point $w \in S$ is an *isolated point* of S if there exists $r > 0$ such that $S \cap D(w, r) = \{w\}$.

Theorem (principle of isolated zeroes)**.** Let $f : D(a, R) \to \mathbb{C}$ be holomorphic and not identically zero. Then there exists $r \in (0, R)$ such that $f(z) \neq 0$ whenever $0 < |z - a| < r$.

Remark. If $f(a) = 0$, then $\{z : f(z) = 0\}$ intersects $D(a, r)$ only at a. Hence, a is an isolated point of the set of zeroes. For instance, there exists no nonzero holomorphic function that vanishes on a line segment or a disc.

We can show that certain identities from real analysis hold for complex functions. For instance, consider the function $g(z) = \sin^2 z + \cos^2 z - 1$. Since this g is holomorphic and vanishes on the real line, g must be identically zero in the complex plane.

The zero set may have an *accumulation point* on the boundary of the domain of f. Consider $f(z) = \sin \frac{1}{z}$ for $z \in D(1, 1)$. Here, if $a_n = \frac{1}{2n\pi}$, then $a_n \in D(1, 1)$ and $f(a_n) = 0$ and $a_n \to 0 \in \partial D(1, 1)$.

Proof. If $f(a) \neq 0$, then by continuity of f there exists $r > 0$ such that $f(z) \neq 0$ for all $z \in D(a, r)$. If $f(a) = 0$, then there exists an integer $m \geq 1$ such that $f(z) = (z-a)^m g(z)$ for $z \in D(a, R)$, where g is holomorphic with $g(a) \neq 0$. In this case, we find that there exists $r > 0$ such that $g(z) \neq 0$ for $z \in D(a, r)$ and hence $f(z) \neq 0$ for $z \in D(a, r) \setminus \{a\}$. $\square$

2.9 Analytic continuation

Theorem. Let $U \subset V$ be domains. If $g_1, g_2 : V \to \mathbb{C}$ are analytic and $g_1 = g_2$ on U, then $g_1 = g_2$ on V. Equivalently, if $f : U \to \mathbb{C}$ is analytic, then there is at most one analytic function $g : V \to \mathbb{C}$ such that $g = f$ on U. We say that g is the *analytic continuation* of f to V, if it exists.

Proof. Let $g_1, g_2 : V \to \mathbb{C}$ be analytic with $g_1|_U = g_2|_U$. Then, $h = g_1 - g_2 : V \to \mathbb{C}$ is analytic, and $h|_U \equiv 0$. We want to show that $h \equiv 0$. Let

$$V_0 = \left\{ z \in V : \exists r > 0, h\Big|_{D(z,r)} \equiv 0 \right\}$$

and

$$V_1 = \left\{ z \in V : \exists n \geq 0, h^{(n)}(z) \neq 0 \right\}$$

Let $z \in V$ and suppose that $z \notin V_0$. Then for any disc $D = D(z,r) \subset V$, we have $h \not\equiv 0$ in D. Hence, by Taylor series, $h^{(n)}(z) \neq 0$ for some n, so $z \in V_1$. Thus, $V = V_0 \cup V_1$. We also know that $V_0 \cap V_1 = \varnothing$.

Note that V_0 is open by definition, and V_1 is by continuity of the derivatives $h^{(n)}$. By connectedness of the domain V, either V_0 or V_1 is empty. Since $U \subset V_0$, we must have $V_1 = \varnothing$. Thus, $V = V_0$ so $h \equiv 0$. □

Remark. The above proof does not rely on holomorphicity but on analyticity. Thus, the theorem holds for real analytic functions. For example, due to *elliptic regularity* (see Part II Analysis of Functions), we can show that harmonic functions are real analytic, and hence have a unique analytic continuation if one exists.

Given a holomorphic function f defined on a disc, we can compute the largest domain containing the disc to which there exists an analytic continuation of f. This is nontrivial to answer in general.

Example. Let $f(z) = \sum_{n=0}^{\infty} z^n$. The radius of convergence of this series is 1, so f is analytic in $D(0,1)$, and there is no larger *disc* $D(0,r) \supset D(0,1)$ such that g has an analytic continuation to $D(0,r)$. However, since $f(z) = \frac{1}{1-z}$ for $z \in D(0,1)$ and the function $\frac{1}{1-z}$ is analytic in $\mathbb{C} \setminus \{1\}$, f indeed has an analytic continuation to the larger domain $\mathbb{C} \setminus \{1\}$.

Example. Let $f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n}$. This function also has a radius of convergence of 1, so f is analytic on $D(0,1)$. It has analytic continuation $\operatorname{Log}(1+z)$ to the domain $\mathbb{C} \setminus \{x \in \mathbb{R} : x \leq -1\}$ containing $D(0,1)$.

Example. Let $f(z) = \sum_{n=0}^{\infty} z^{n!}$. This has radius of convergence 1, so f is analytic in $D(0,1)$. However, f has no analytic continuation to any larger domain containing $D(0,1)$. The boundary $\partial D(0,1)$ is known as the *natural boundary* of f.

We can find in fact that for any given domain $U \subset \mathbb{C}$, there exists a holomorphic function $f : U \to \mathbb{C}$ which has no analytic continuation to a domain properly containing U.

The failure of analytic continuation in some cases can be explained as the result of loss of a regularity condition, such as boundedness, continuity, differentiability, or so on. However, this is not always the reason, and analytic continuation may remain impossible even when regularity conditions are all satisfied.

Example. Let $f(z) = \sum_{n=0}^{\infty} \exp\left(-2^{n/2}\right) z^{2^n}$, which has unit radius of convergence. f, and its derivatives of any order, are uniformly continuous on the closed disc $\overline{D(0,1)}$. However, we can prove that it has natural boundary $\partial D(0,1)$, using the following theorem which will not be proven.

Theorem (Ostrowski–Hadamard gap theorem). Let (p_n) be a sequence of positive integers such that $p_{n+1} > (1+\delta)p_n$ for all n and some fixed $\delta > 0$. If (c_n) is a sequence of complex numbers such that $f(z) = \sum_{n=0}^{\infty} c_n z^{p_n}$ has unit radius of convergence, then $\partial D(0,1)$ is the natural boundary of f.

Corollary (identity principle). Let $f, g : U \to \mathbb{C}$ be holomorphic functions in a domain U. If the set $S = \{z \in U : f(z) = g(z)\}$ contains a non-isolated point, then $f = g$ in U.

Proof. Let $h = f - g$, so $S = \{z \in U : h(z) = 0\}$. Suppose that S has a non-isolated point w. If there exists $r > 0$ such that $h \not\equiv 0$ in $D(w, r)$, then by the principle of isolated zeroes, we can find $\varepsilon > 0$ such that $f(z) \neq 0$ whenever $0 < |z - w| < \varepsilon$. However, this contradicts the assumption that w is a non-isolated point of S. Thus, $h \equiv 0$ on $D(w, r)$ for all $D(w, r) \subset U$. Thus, $h \equiv 0$ on U, so $f = g$ on U. □

Corollary (global maximum principle). Let U be a bounded open set. Suppose $f : \overline{U} \to \mathbb{C}$ is a continuous function such that f is holomorphic in U. Then $|f|$ attains its maximum on ∂U.

Proof. $\overline{U}$ is compact, and $|f|$ is continuous on $\overline{U}$. Hence, it attains its maximum; there exists $w \in \overline{U}$ such that $|f(w)| = \max_{z \in \overline{U}} |f(z)|$. If $w \notin U$, then $w \in \partial U$ as required. Otherwise, let $D = D(w, r) \subset U$. Since $|f(z)| \leq |f(w)|$ for all $z \in D$, the local maximum principle implies that f is constant on D. Hence, by the identity principle, f is constant on the connected component of U containing D, which will be written U'. By continuity, f is constant on the closure of this connected component $\overline{U'}$. In particular, $|f(z)| = |f(w)|$ for all $z \in \partial U' \subseteq \partial U$ as required. □

Theorem (Cauchy's integral formula for derivatives). Let $f : D(a, R) \to \mathbb{C}$ be holomorphic. For any $\rho \in (0, R)$ and $w \in D(a, \rho)$, we have

$$f^{(k)}(w) = \frac{k!}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)\,\mathrm{d}z}{(z - w)^{k+1}}$$

Further,

$$\sup_{z \in D(a, R/2)} \left| f^{(k)}(z) \right| \leq \frac{C}{R^k} \sup_{z \in D(a,R)} |f(z)|$$

where $C = k! 2^{k+1}$ is a constant which depends only on k. This final result is called a Cauchy estimate for the kth derivative.

Remark. Directly applying Cauchy's integral formula to $f^{(n)}$, we find a formula for $f^{(n)}(w)$ in terms of an integral involving $f^{(n)}$. The significance of the above theorem is that the integral involves f alone, and not its derivatives.

Note that we have already observed the special case $w = a$. This was proven during the discussion on Taylor series.

Proof. If $k = 0$, we have the usual Cauchy integral formula. For $k = 1$, let $g(z) = \frac{f(z)}{z-w}$, which is holomorphic in $D(a,R) \setminus \{w\}$, with derivative

$$g'(z) = \frac{f'(z)}{z-w} - \frac{f(z)}{(z-w)^2}$$

Since $\partial D(a,\rho) \subset D(a,R) \setminus \{w\}$, we know that $\int_{\partial D(a,\rho)} g'(z)\,\mathrm{d}z = 0$ by the fundamental theorem of calculus. Applying the usual Cauchy integral formula to f',

$$f'(w) = \frac{1}{2\pi i}\int_{\partial D(a,\rho)} \frac{f'(z)\,\mathrm{d}z}{z-w}$$

Combining these results give the result for $k=1$. For higher derivatives, we can proceed by induction. Let $k \geq 2$, and then suppose the formula holds for this value of k, for all holomorphic functions $D(a,R) \to \mathbb{C}$. For any holomorphic function $f : D(a,R) \to \mathbb{C}$, consider

$$g(z) = \frac{f(z)}{(z-w)^{k+1}} \implies g'(z) = \frac{f'(z)}{(z-w)^{k+1}} - \frac{(k+1)f(z)}{(z-w)^{k+2}}$$

which is defined in $D(a,R) \setminus \{w\}$. Then, since $\int_{\partial D(a,\rho)} g'(z)\,\mathrm{d}z = 0$, we find

$$\int_{\partial D(a,\rho)} \frac{f'(z)\,\mathrm{d}z}{(z-w)^{k+1}} = (k+1)\int_{\partial D(a,\rho)} \frac{f(z)\,\mathrm{d}z}{(z-w)^{k+2}}$$

By substituting f' into the induction hypothesis,

$$f^{(k+1)}(w) = \frac{k!}{2\pi i}\int_{\partial D(a,\rho)} \frac{f'(z)\,\mathrm{d}z}{(z-w)^{k+1}}$$

We can then combine the previous two expressions to find

$$f^{(k+1)}(w) = \frac{(k+1)!}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(z)\,\mathrm{d}z}{(z-w)^{k+2}}$$

as required.

For the second part, let $\sup_{z\in D(a,R)} |f(z)| < \infty$ without loss of generality. Let $\rho \in (R/2, R)$. Then, by the first part, for all $w \in D(a,R/2)$ we have

$$\left|f^{(k)}(w)\right| \leq \frac{k!}{2\pi}\left(\sup_{z\in\partial D(a,\rho)} \frac{|f(z)|}{|z-w|^{k+1}}\right)\mathrm{length}(\partial D(a,\rho))$$

As $|z-w| \geq \rho - R/2$ for all $z \in \partial D(a,\rho)$ and all $w \in D(a,R/2)$, this implies

$$\sup_{w\in D(a,R/2)} \left|f^{(k)}(w)\right| \leq \frac{k!\rho}{(\rho - R/2)^{k+1}} \sup_{z\in D(a,R)} |f(z)|$$

Now, as $\rho \to R$, the result follows. □

2.10 Uniform limits of holomorphic functions

Definition. Let $U \subseteq \mathbb{C}$ be open, and let $f_n : U \to \mathbb{C}$ be a sequence of functions. We say that (f_n) converges *locally uniformly* on U if, for all $a \in U$, there exists $r > 0$ such that (f_n) converges uniformly on $D(a, r)$.

Example. Let $f_n(z) = z^n$. Then $f_n \to 0$ locally uniformly, but not uniformly.

Proposition. (f_n) converges locally uniformly on an open set $U \subseteq \mathbb{C}$ if and only if (f_n) converges uniformly on each compact subset $K \subseteq U$.

Proof. The forward implication is simple, due to the definition of compactness. The converse follows since for all $a \in U$, there exists a compact disc $\overline{D(a, r)} \subset U$. □

Theorem (uniform limits of holomorphic functions)**.** Let $U \subseteq \mathbb{C}$ be open, and $f_n : U \to \mathbb{C}$ be holomorphic for each $n \in \mathbb{N}$. If (f_n) converges locally uniformly on U to some function $f : U \to \mathbb{C}$, then f is holomorphic.
Further, $f_n' \to f'$ locally uniformly on U, and by induction, for each k we have $f_n^{(k)} \to f^{(k)}$ locally uniformly on U as $n \to \infty$.

Remark. This is not true for real analytic functions. The *Weierstrass approximation theorem* states the following. Let $f : [a, b] \to \mathbb{R}$ be a continuous function on a compact interval $[a, b] \subset \mathbb{R}$. Then, there exists a sequence of polynomials (p_n) converging uniformly to f on $[a, b]$.

There exist continuous, nowhere differentiable functions $f : [a, b] \to \mathbb{R}$. Applying the Weierstrass approximation theorem to such functions f shows that the uniform limit of real analytic functions need not have a single point of differentiability.

Proof. Let $a \in U$ and choose $r > 0$ such that $\overline{D(a, r)} \subset U$ and $f_n \to f$ uniformly on $\overline{D(a, r)}$. Since the f_n are continuous, by a result from Analysis and Topology we have that f is continuous in $\overline{D(a, r)}$.

Let γ be a closed curve in $D(a, r)$. Since $D(a, r)$ is convex, by the convex Cauchy theorem we have $\int_\gamma f_n(z) \, \mathrm{d}z = 0$. Since $f_n \to f$ uniformly on $D(a, r)$, it follows that

$$\int_\gamma f(z) \, \mathrm{d}z = \lim_{z \to \infty} \int_\gamma f_n(z) \, \mathrm{d}z = 0$$

By Morera's theorem, f is holomorphic in $D(a, r)$. Since a is arbitrary, f is holomorphic on all of U.

Now, let $a \in U$ be arbitrary and let $D(a, r)$ be as above. We can apply the Cauchy estimate for $k = 1$, $R = r$, applied to the function $f_n - f$. This gives

$$\sup_{z \in D(a, r/2)} |f_n'(z) - f'(z)| \leq \frac{4}{r} \sup_{z \in D(a, r)} |f_n(z) - f(z)|$$

Since the right hand side converges to zero as $n \to \infty$, the claim follows. □

Remark. Many of the key results proven for holomorphic functions have analogues for real harmonic functions on domains not just in $\mathbb{R}^2$ but in $\mathbb{R}^n$ for any n. For instance:

(i) (Liouville's theorem) if $u: \mathbb{R}^n \to \mathbb{R}$ is a bounded harmonic function then u is constant;

(ii) (local maximum principle) if $u: D(a,r)$ is a C^2 harmonic function on an open ball $D(a,r)$ in $\mathbb{R}^n$, and if $u(x) \leq u(a)$ for all $x \in D(a,r)$, then u is constant;

(iii) (global maximum principle) a harmonic function on a bounded open set U that is continuous on $\overline{U}$ attains its maximum on ∂U;

(iv) harmonic functions are real analytic;

(v) the unique analytic continuation principle holds;

(vi) uniform limits of harmonic functions are harmonic;

(vii) derivative estimates hold: if $u: D(a,R) \subseteq \mathbb{R}^n \to \mathbb{R}$ is harmonic, then

$$\sup_{D(a,R/2)} |D^k u| \leq CR^{-k} \sup_{D(a,R)} |u|; \quad C = C(n,k)$$

For the case $n = 2$, the result for harmonic functions can often be deduced directly from the corresponding results for holomorphic functions. For instance, for Liouville's theorem, given that u is a harmonic function on $\mathbb{R}^2$, we find a function v such that $u + iv$ is holomorphic on $\mathbb{C}$ (which is always possible in a simply connected domain). Then $g = e^f$ is holomorphic with $|g| = e^u$, so if u is bounded then g is bounded. By Liouville's theorem for holomorphic functions, g and hence f is constant.

3 More integration

3.1 Winding numbers

Let $\gamma: [a,b] \to \mathbb{C}$ be a closed, piecewise C^1 curve, and let $w \notin \operatorname{Im}\gamma$. For all t, there exists $r(t) > 0$ and $\theta(t) \in \mathbb{R}$ such that $\gamma(t) = w + r(t)e^{i\theta(t)}$. Then, the function $r: [a,b] \to \mathbb{R}$ is given by $r(t) = |\gamma(t) - w|$, so it is uniquely determined and piecewise C^1.

Definition. If we have a continuous choice of $\theta: [a,b] \to \mathbb{R}$ such that $\gamma(t) = w + r(t)e^{i\theta(t)}$, then we define the *winding number* or the *index* of γ about w as

$$I(\gamma; w) = \frac{\theta(b) - \theta(a)}{2\pi}$$

If γ is a closed curve, $I(\gamma; w)$ is an integer. This is because

$$\gamma(a) = \gamma(b) \implies \exp(i\theta(b) - i\theta(a)) = 1$$

If $\theta_1: [a,b] \to \mathbb{C}$ is also continuous such that $\gamma(t) = w + re^{i\theta_1(t)}$, then $\exp(i\theta(t) - i\theta_1(t)) = 1$, so

$$\frac{\theta_1(t) - \theta(t)}{2\pi} \in \mathbb{Z}$$

Since $\theta_1 - \theta$ is continuous, this quotient must be a constant. Hence, $I(\gamma; w)$ is well-defined and independent of the (continuous) choice of θ.

Lemma. Let $w \in \mathbb{C}$ and $\gamma : [a, b] \to \mathbb{C} \setminus \{w\}$, where γ is piecewise C^1. Then, there exists a piecewise C^1 function $\theta : [a, b] \to \mathbb{R}$ such that $\gamma(t) = w + r(t)e^{i\theta(t)}$, where $r(t) = |\gamma(t) - w|$. If γ is closed, then we also have

$$I(\gamma; w) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - w}$$

Remark. If γ is C^1, and there is a C^1 function θ such that $\gamma(t) = w + r(t)e^{i\theta(t)}$, then

$$\gamma'(t) = (r'(t) + ir(t)\theta'(t))e^{i\theta(t)} = \left(\frac{r'(t)}{r(t)} + i\theta'(t)\right) r(t)e^{i\theta(t)} = \left(\frac{r'(t)}{r(t)} + i\theta'(t)\right)(\gamma(t) - w)$$

Hence,

$$\theta'(t) = \operatorname{Im} \frac{\gamma'(t)}{\gamma(t) - w} \implies \theta(t) = \theta(a) + \operatorname{Im} \int_a^t \frac{\gamma'(s)\,\mathrm{d}s}{\gamma(s) - w}$$

Proof. Let $h(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s)-w}\,\mathrm{d}s$. The integrand is bounded on $[a, b]$, and is continuous except at the finite number of points at which γ' may be discontinuous. Hence, $h : [a, b] \to \mathbb{C}$ is continuous. Further, h is differentiable with $h'(t) = \frac{\gamma'(t)}{\gamma(t)-w}$ at each t where γ' is continuous. Hence, h is piecewise C^1. This induces an ordinary differential equation for $\gamma(t) - w$.

$$(\gamma(t) - w)' - (\gamma(t) - w)h'(t) = 0$$

which is true for all $t \in [a, b]$ except possibly for a finite set. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((\gamma(t) - w)e^{-h(t)}\right) = \gamma'(t)e^{-h(t)} - (\gamma(t) - w)e^{-h(t)}h'(t) = 0$$

except for finitely many t. Since $(\gamma(t) - w)e^{-h(t)}$ is continuous, it must be constant, and equal to its value at $t = a$. Hence

$$\gamma(t) - w = (\gamma(a) - w)e^{h(t)} = (\gamma(a) - w)e^{\operatorname{Re} h(t)}e^{i \operatorname{Im} h(t)} = |\gamma(a) - w|e^{\operatorname{Re} h(t)}e^{i(\alpha + \operatorname{Im} h(t)}$$

for α such that $e^{i\alpha} = \frac{\gamma(a)-w}{|\gamma(a)-w|}$. Hence, we can set $\theta(t) = \alpha + \operatorname{Im} h(t)$.

For the second part, note that

$$I(\gamma; w) = \frac{\theta(b) - \theta(a)}{2\pi} = \frac{\operatorname{Im}(h(b) - h(a))}{2\pi} = \frac{\operatorname{Im} h(b)}{2\pi}$$

Since $\gamma(t) - w = (\gamma(a) - w)e^{h(t)}$ and $\gamma(b) = \gamma(a)$, we have $e^{h(b)} = 1$, so $\operatorname{Re} h(b) = 0$ and $\operatorname{Im} h(b) = -ih(b)$. Thus,

$$I(\gamma; w) = \frac{1}{2\pi i} h(b) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(s)}{\gamma(s) - w}\,\mathrm{d}s = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - w}$$

$\square$

Remark. It is also true that θ exists and is continuous if γ is merely continuous, but the formula for the winding number is not useful, so we omit this proof.

Proposition. If $\gamma : [a, b] \to \mathbb{C}$ is a closed curve, then the function $w \mapsto I(\gamma; w)$ is continuous on $\mathbb{C} \setminus \operatorname{Im} \gamma$. Since $I(\gamma; w)$ is integer-valued, $I(\gamma; w)$ is locally constant. So $I(\gamma; w)$ is constant for each connected component of the open set $\mathbb{C} \setminus \operatorname{Im} \gamma$.

Proof. Exercise. □

Proposition. If $\gamma : [a, b] \to D(z_0, R)$ is a closed curve, then $I(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus D(z_0, R)$.
If $\gamma : [a, b] \to \mathbb{C}$ is a closed curve, then there exists a unique unbounded connected component Ω of $\mathbb{C} \setminus \gamma([a, b])$, and $I(\gamma; w) = 0$ for all $w \in \Omega$.

Proof. For the first part, if $w \in \mathbb{C} \setminus D(z_0, R)$, then the function $f(z) = \frac{1}{z-w}$ is holomorphic in $D(z_0, R)$. Hence $I(\gamma; w) = 0$ by the convex version of Cauchy's theorem.

For the second part, since $\gamma([a, b])$ is compact (by continuity of γ), there exists $R > 0$ such that $\gamma([a, b]) \subset D(0, R)$. Since $\mathbb{C} \setminus D(0, R)$ is a connected subset of $\mathbb{C} \setminus \gamma([a, b])$, there exists a connected component Ω of $\mathbb{C} \setminus \gamma([a, b])$ such that $\mathbb{C} \setminus D(0, R) \subseteq \Omega$. This component is unbounded. Any other component is disjoint from $\mathbb{C} \setminus D(0, R)$, so is contained within $D(0, R)$ and is hence bounded. So the unbounded component is unique. Since $I(\gamma; w)$ is locally constant and zero on $\mathbb{C} \setminus D(0, R)$, it is zero on Ω. □

3.2 Continuity of derivative function

Lemma. Let $f : U \to \mathbb{C}$ be holomorphic, and define $g : U \times U \to \mathbb{C}$ by

$$g(z, w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w \end{cases}$$

Then g is continuous. Moreover, if γ is a closed curve in U, then the function $h(w) = \int_\gamma g(z, w)\,\mathrm{d}z$ is holomorphic on U.

Proof. It is clear that g is continuous at (z, w) if $z \neq w$. To check continuity at a point $(a, a) \in U \times U$, let $\varepsilon > 0$ and choose $\delta > 0$ such that $D(a, \delta) \subseteq U$ and $|f'(z) - f'(a)| < \varepsilon$ for all $z \in D(a, \delta)$. This is always possible since f' is continuous.

Let $z, w \in D(a, \delta)$. If $z = w$, then $|g(z, w) - g(a, a)| = |f'(z) - f'(a)| < \varepsilon$. If $z \neq w$, we have $tz + (1 - t)w \in D(a, \delta)$ for $t \in [0, 1]$. Hence,

$$\begin{aligned} f(z) - f(w) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(tz + (1 - t)w)\,\mathrm{d}t \\ &= \int_0^1 f'(tz + (1 - t)w)(z - w)\,\mathrm{d}t \\ &= (z - w) \int_0^1 f'(tz + (1 - t)w)\,\mathrm{d}t \end{aligned}$$

Thus,

$$\begin{aligned}|g(z,w)-g(a,a)| &= \left|\frac{f(z)-f(w)}{z-w} - f'(a)\right| \\ &= \left|\int_0^1 [f'(tz+(1-t)w) - f'(a)]\,\mathrm{d}t\right| \\ &\leq \sup_{t\in[0,1]} |f'(tz+(1-t)w) - f'(a)| < \varepsilon\end{aligned}$$

Hence $|(z,w)-(a,a)| < \delta$ implies $|g(z,w)-g(a,a)| < \varepsilon$, so g is continuous at (a,a).

To show h is holomorphic, we must first check that h is continuous. Let $w_0 \in W$, and suppose $w_n \to w_0$. Let $\delta > 0$ such that $\overline{D(w_0,\delta)} \subset U$. The function g is continuous on $U\times U$, so it is uniformly continuous on the compact subset $\operatorname{Im}\gamma \times \overline{D(w_0,\delta)} \subset U \times U$. Thus, if we let $g_n(z) = g(z,w_n)$ and $g_0(z) = g(z,w_0)$ for $z \in \operatorname{Im}\gamma$, then $g_n \to g_0$ uniformly on $\operatorname{Im}\gamma$. Hence $\int_\gamma g_n(z)\,\mathrm{d}z \to \int_\gamma g_0(z)\,\mathrm{d}z$. In other words, $h(w_n) \to h(w_0)$. Thus, h is continuous.

Now, we can use the convex Cauchy's theorem and Morera's theorem to show h is holomorphic on U. For $w_0 \in U$, we can choose a disc $D(w_0,\delta) \subset U$. Suppose that γ is parametrised over $[a,b]$, and let $\beta\colon [c,d] \to D(w_0,\delta)$ be any closed curve. Then $h(w) = \int_\gamma g(z,w)\,\mathrm{d}z = \int_a^b g(\gamma(t),w)\gamma'(t)\,\mathrm{d}t$, hence

$$\begin{aligned}\int_\beta h(w)\,\mathrm{d}w &= \int_c^d \left(\int_a^b g(\gamma(t),\beta(s))\gamma'(t)\beta'(s)\,\mathrm{d}t\right)\mathrm{d}s \\ &= \int_a^b \left(\int_c^d g(\gamma(t),\beta(s))\gamma'(t)\beta'(s)\,\mathrm{d}s\right)\mathrm{d}t \\ &= \int_\gamma \left(\int_\beta g(z,w)\,\mathrm{d}w\right)\mathrm{d}z\end{aligned}$$

by Fubini's theorem, which will be proven below. By a previous theorem, for all $z \in U$, the function $w \mapsto g(z,w)$ is holomorphic in $D(w_0,\delta)$ (and hence in U), since it is continuous in U and holomorphic except at a single point z. Hence, by the convex version of Cauchy's theorem, $\int_\beta g(z,w)\,\mathrm{d}w = 0$. Hence, $\int_\beta h(w)\,\mathrm{d}w = 0$. By Morera's theorem, h is holomorphic in $D(w_0,\delta)$ and hence on U. □

Lemma (Fubini's theorem)**.** If $\varphi\colon [a,b]\times[c,d] \to \mathbb{R}$ is continuous, then $f_1\colon s \mapsto \int_c^d \varphi(s,t)\,\mathrm{d}t$ is continuous on $[a,b]$, the function $f_2\colon t \mapsto \int_a^b \varphi(s,t)\,\mathrm{d}t$ is continuous on $[c,d]$, and

$$\int_a^b \left(\int_c^d \varphi(s,t)\,\mathrm{d}t\right)\mathrm{d}s = \int_c^d \left(\int_a^b \varphi(s,t)\,\mathrm{d}s\right)\mathrm{d}t$$

Proof. Since φ is continuous on the compact set $[a,b]\times[c,d]$, it is uniformly continuous. Hence, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|s_1 - s_2| < \delta \implies |\varphi(s_1,t) - \varphi(s_2,t)| < \varepsilon$ for all $t \in [c,d]$, so $|f_1(s_1) - f_1(s_2)| < (d-c)\varepsilon$, so f_1 is continuous. Similarly, f_2 is continuous. Note that since φ is uniformly continuous, it is the uniform limit of a sequence of step functions of the form $g(x,y) = \sum_{j=1}^N \alpha_j \chi_{R_j}(x,y)$ where α_j are constants, and R_j are sub-rectangles of the form $R_j = [a_j,b_j)\times[c_j,d_j)$ such that $\bigcup R_j$ is a finite partition of $[a,b)\times[c,d)$, and χ_{R_j} is the characteristic function of R_j For such step functions, we can easily check the interchangability of the integrals. □

3.3 Cauchy's theorem and Cauchy's integral formula

Definition. Let $U \subseteq \mathbb{C}$ be open. A closed curve $\gamma\colon [a,b] \to U$ is said to be *homologous to zero* in U if $I(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus U$.

Theorem. Let U be a non-empty open subset of $\mathbb{C}$, and γ be a closed curve in U homologous to zero in U. Then,

$$I(\gamma; w) f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)\,\mathrm{d}z}{z - w}$$

for every holomorphic function $f\colon U \to \mathbb{C}$ and every $w \in U \setminus \operatorname{Im}\gamma$. Further,

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

for every holomorphic $f\colon U \to \mathbb{C}$.

Remark. Cauchy's theorem states that if $\int_\gamma f(z)\,\mathrm{d}z = 0$ for a specific family of holomorphic functions on U, namely for $f_w(z) = \frac{1}{z-w}$ where $w \in \mathbb{C} \setminus U$, then $\int_\gamma f(z)\,\mathrm{d}z = 0$ for any holomorphic function $f\colon U \to \mathbb{C}$.

The first and second parts as statements are equivalent. Indeed, if we assume the Cauchy integral formula holds, simply apply the formula with $F(z) = (z-w)f(z)$. Since $F(w) = 0$, we have $\int_\gamma f(z)\,\mathrm{d}z = 0$. If we assume Cauchy's theorem, for any $w \in U$, the function

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w \end{cases}$$

is holomorphic in U as seen above. Hence $\int_\gamma g(z)\,\mathrm{d}z = 0$, so $\frac{1}{2\pi i}\int_\gamma \frac{f(z)\mathrm{d}z}{z-w} = I(\gamma; w)f(w)$ for all $w \notin \operatorname{Im}\gamma$.

Note that the statement that γ is homologous to zero is equivalent to Cauchy's theorem being valid for all f. For example, given $w \in \mathbb{C} \setminus U$, we can apply Cauchy's theorem to $f(z) = \frac{1}{z-w}$ to get $I(\gamma; w) = 0$. The converse is proven in the theorem following this proof. This is also equivalent to Cauchy's integral formula being valid for all f.

Proof. It suffices to prove part (i). Equivalently, for all $w \in U \setminus \operatorname{Im}\gamma$,

$$\int_\gamma \frac{f(z) - f(w)}{z - w}\,\mathrm{d}z = 0 \iff \int_\gamma g(z, w)\,\mathrm{d}z = 0$$

where

$$g(z, w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w \end{cases}$$

Now, define

$$h\colon U \to \mathbb{C}; \quad h(w) = \int_\gamma g(z, w)\,\mathrm{d}z$$

By the above lemma, this is holomorphic on U. We will show that $h = 0$. We will extend h to a holomorphic function $H\colon \mathbb{C} \to \mathbb{C}$ and prove that $H(w) \to 0$ as $w \to \infty$, then we can apply Liouville's theorem.

To extend h into an entire function H, by definition of γ being homologous to zero in U, we have $\mathbb{C} \setminus U \subseteq V \equiv \{w \in \mathbb{C} \setminus \operatorname{Im}\gamma\colon I(\gamma; w) = 0\}$. So $\mathbb{C} = U \cup V$, and V is open since $I(\gamma;\,\cdot\,)$ is locally constant. For $w \in U \cap V$, we have

$$h(w) = \int_\gamma \frac{f(z) - f(w)}{z - w}\,\mathrm{d}z = \int_\gamma \frac{f(z)\,\mathrm{d}z}{z - w}$$

since $\int_\gamma \frac{\mathrm{d}z}{z-w} = 2\pi i \cdot I(\gamma; w) = 0$ as $w \in V$. Hence, on $U \cap V$, the function h agrees with

$$h_1\colon V \to \mathbb{C};\quad h_1(w) = \int_\gamma \frac{f(z)\,\mathrm{d}z}{z - w}$$

We know that h_1 is holomorphic on V. Hence, the function $H\colon \mathbb{C} \to \mathbb{C}$ defined by

$$H(w) = \begin{cases} h(w) & w \in U \\ h_1(w) & w \in V \end{cases}$$

is well-defined and holomorphic.

Now, we will show $H(w) \to 0$ as $|w| \to \infty$. Let $R > 0$ such that $\operatorname{Im}\gamma \subset D(0, R)$, which is possible since $\operatorname{Im}\gamma$ is compact. Hence, $\mathbb{C} \setminus D(0, R) \subseteq V$. If $|w| > R$,

$$|H(w)| = |h_1(w)| = \left|\int_\gamma \frac{f(z)\,\mathrm{d}z}{z - w}\right| \leq \frac{1}{|w| - R}\left(\sup_{z \in \operatorname{Im}\gamma} |f(z)|\right)\operatorname{length}(\gamma)$$

Hence, $H(w) \to 0$ as $|w| \to \infty$, as claimed. Hence H is bounded, since H is continuous, and $|H(w)| \leq 1$ outside some closed disc $\overline{D(0, R_1)}$. By Liouville's theorem, H is constant, and by the claim, $H = 0$. In particular, $h = 0$. □

Corollary. Let $U \subset \mathbb{C}$ be open and $\gamma_1, \dots, \gamma_n$ be closed curves in U such that $\sum_{j=1}^n I(\gamma_j; w) = 0$ for all $w \in \mathbb{C} \setminus U$. Then, for any holomorphic $f\colon U \to \mathbb{C}$, we have

$$f(w) \sum_{j=1}^n I(\gamma_j; w) = \sum_{j=1}^n \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)\,\mathrm{d}z}{z - w}$$

for all $w \in U \setminus \bigcup_{j=1}^n \operatorname{Im}\gamma_j$, and

$$\sum_{j=1}^n \int_{\gamma_j} f(z)\,\mathrm{d}z = 0$$

Proof. For the first part, define $g(z, w)$ as before, but let

$$V = \left\{ w \in \mathbb{C} \setminus \bigcup_{j=1}^n \operatorname{Im}\gamma_j \colon \sum_{j=1}^n I(\gamma_j; w) = 0 \right\}$$

In the definitions of h and h_1, use the sum of the integrals over γ_j. Then we can proceed as above. The second part follows from the first as before. □

Corollary. Let $U \subset \mathbb{C}$ be open and let β_1, β_2 be closed curves in U such that $I(\beta_1; w) = I(\beta_2; w)$ for all $w \in \mathbb{C} \setminus U$. Then

$$\int_{\beta_1} f(z)\,\mathrm{d}z = \int_{\beta_2} f(z)\,\mathrm{d}z$$

for all holomorphic functions $f : U \to \mathbb{C}$.

Proof. We can apply the second part of the previous corollary with $\gamma_1 = \beta_1$ and $\gamma_2 = -\beta_2$, noting that $I(-\beta_2; w) = -I(\beta_2; w)$ for any $w \notin \operatorname{Im} \beta_2$. □

3.4 Homotopy

The set of closed curves in U such that Cauchy's theorem is valid is the set of holomorphic functions homologous to zero. We will now construct a more restrictive condition, the condition of being *null-homotopic*.

Definition. Let $U \subseteq \mathbb{C}$ be a domain, and let $\gamma_0, \gamma_1 : [a, b] \to U$ be closed curves. We say that γ_0 is *homotopic to* γ_1 *in* U if there exists a continuous map $H : [0, 1] \times [a, b] \to U$ such that for all $s \in [0, 1]$, $t \in [a, b]$,

$$H(0, t) = \gamma_0(t); \quad H(1, t) = \gamma_1(t); \quad H(s, a) = H(s, b)$$

Such a map is called a *homotopy* between γ_0, γ_1.

For $0 \le s \le 1$, if we let $\gamma_s : [a, b] \to U$ be defined by $\gamma_s(t) = H(s, t)$ for $t \in [a, b]$, then the above conditions imply that $\{\gamma_s : s \in [0, 1]\}$ is a family of continuous closed curves in U which deform γ_0 to γ_1 continuously without leaving U.

Definition. A closed curve is *null-homotopic* in a certain domain if it is homotopic to a constant curve in the domain, such as $\gamma(t) = z$ for z fixed.

Theorem. If $\gamma_0, \gamma_1 : [a, b] \to U$ are homotopic closed curves in U, then $I(\gamma_0; w) = I(\gamma_1; w)$ for all $w \in \mathbb{C} \setminus U$. In particular, if a closed curve γ is null-homotopic in U, it is homologous to zero in U.

Proof. Let $H : [0, 1] \times [a, b] \to U$ be a homotopy between γ_0 and γ_1. Since H is continuous and $[0, 1] \times [a, b]$ is compact, the image $K = H([0, 1] \times [a, b])$ is a compact subset of the open set U. Therefore, there exists $\varepsilon > 0$ such that for all $w \in \mathbb{C} \setminus U$, $|w - H(s, t)| > 2\varepsilon$ for all $(s, t) \in [0, 1] \times [a, b]$. Since H is uniformly continuous on $[0, 1] \times [a, b]$, there exists $n \in \mathbb{N}$ such that

$$\forall (s, t), (s', t') \in [0, 1] \times [a, b],\ |s - s'| + |t - t'| \le \frac{1}{n} \implies |H(s, t) - H(s', t')| < \varepsilon$$

For $k = 0, 1, 2, \dots, n$, we let $\Gamma_k(t) = H(k/n, t)$ for $a \le t \le b$. Then the Γ_k are closed continuous curves with $\Gamma_0 = \gamma_0$ and $\Gamma_n = \gamma_1$. Hence, for all $t \in [a, b]$,

$$\underbrace{|\Gamma_{k-1}(t) - \Gamma_k(t)|}_{<\varepsilon} < \underbrace{|w - \Gamma_{k-1}(t)|}_{>2\varepsilon}$$

On the example sheets we have shown that for piecewise C^1 closed curves $\gamma, \tilde{\gamma}$, if we have $|\gamma(t) - \tilde{\gamma}(t)| < |w - \gamma(t)|$ for all t, then $I(\gamma; w) = I(\tilde{\gamma}; w)$. Hence, if Γ_k are piecewise C^1, we can see that $I(\Gamma_{k-1}; w) = I(\Gamma_k; w)$ for all k, and hence $I(\gamma_0; w) = I(\gamma_1; w)$ as required.

We have only assumed that H is continuous, so Γ_k need not be piecewise C^1. We can fix this problem by approximating each Γ_k by a polygonal curve. We can take

$$\tilde{\Gamma}_k(t) = \left(1 - \frac{n(t - a_{j-1})}{b - a}\right) H\left(\frac{k}{n}, a_{j-1}\right) + \frac{n(t - a_{j-1})}{b - a} H\left(\frac{k}{n}, a_j\right)$$

for $a_{j-1} \le t \le a_j$, where

$$a_j = a + \frac{(b - a)j}{n}$$

If we choose n so that

$$|s - s'| + |t - t'| \le \frac{\min\{1, b - a\}}{n} \implies |H(s, t) - H(s', t')| < \varepsilon$$

the curves $\tilde{\Gamma}_k$ satisfy

$$|\tilde{\Gamma}_{k-1}(t) - \tilde{\Gamma}_k(t)| < |w - \tilde{\Gamma}_{k-1}(t)|$$

for all $t \in [a, b]$. This is because for $t \in [a_{j-1}, a_j]$,

$$\begin{aligned}
|\tilde{\Gamma}_{k-1}(t) - \tilde{\Gamma}_k(t)| &\le \left(1 - \frac{n(t - a_{j-1})}{b - a}\right) \left| H\left(\frac{k-1}{n}, a_{j-1}\right) - H\left(\frac{k}{n}, a_{j-1}\right) \right| \\
&\quad + \frac{n(t - a_{j-1})}{b - a} \left| H\left(\frac{k-1}{n}, a_j\right) - H\left(\frac{k}{n}, a_j\right) \right| \\
&< \varepsilon
\end{aligned}$$

and

$$|w - \tilde{\Gamma}_{k-1}(t)| \ge |w - \Gamma_{k-1}(t)| - |\Gamma_{k-1}(t) - \tilde{\Gamma}_{k-1}(t)| > 2\varepsilon - \varepsilon = \varepsilon$$

We also have, for all $t \in [a, b]$,

$$|\tilde{\Gamma}_0(t) - \gamma_0(t)|; \quad |\tilde{\Gamma}_n - \gamma_1(t)| < |w - \gamma_1(t)|$$

Hence the result follows from the same example sheet question. □

Remark. If γ is homologous to zero in U, it is not necessarily the case that γ is null-homotopic. For insance, let $U = \mathbb{C} \setminus \{w_1, w_2\}$ for $w_1 \ne w_2$, and let $U_1 = U \cup \{w_1\} = \mathbb{C} \setminus \{w_2\}$ and $U_2 = U \cup \{w_2\} = \mathbb{C} \setminus \{w_1\}$. Then, consider a curve γ which is not null-homotopic in U, but null-homotopic in each of the larger domains U_1, U_2. Then γ is homologous to zero in U_1 and U_2. Hence $I(\gamma; w_1) = I(\gamma; w_2) = 0$, so γ is homologous to zero in U.

Corollary. If $\gamma_0, \gamma_1 \colon [a,b] \to U$ are homotopic closed curves in U, then

$$\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$$

for all holomorphic $f \colon U \to \mathbb{C}$.

This is immediate from previous results. However, we can make a direct proof that does not require the most general form of Cauchy's theorem.

Proof. With $\tilde{\Gamma}_k$ as above, consider the closed curve comprised of

(i) the curve $\tilde{\Gamma}_{k-1}$ on $[a_{j-1}, a_j]$;

(ii) the line segment $\left[\tilde{\Gamma}_{k-1}(a_j), \tilde{\Gamma}_k(a_j)\right]$;

(iii) the curve $-\tilde{\Gamma}_k$ on $[a_j, a_{j-1}]$;

(iv) the line segment $\left[\tilde{\Gamma}_k(a_{j-1}), \tilde{\Gamma}_{k-1}(a_{j-1})\right]$.

This curve is contained in the disc $D(\tilde{\Gamma}_{k-1}(a_{j-1}), \varepsilon) \subseteq U$. We can apply the convex version of Cauchy's theorem and sum over j to find

$$\int_{\tilde{\Gamma}_{k-1}} f(z)\,\mathrm{d}z = \int_{\tilde{\Gamma}_k} f(z)\,\mathrm{d}z$$

Similarly we can find

$$\int_{\tilde{\Gamma}_0} f(z)\,\mathrm{d}z = \int_{\gamma_0} f(z)\,\mathrm{d}z\,; \quad \int_{\tilde{\Gamma}_n} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$$

□

3.5 Simply connected domains

Definition. A domain U is *simply connected* if every closed curve in u is null-homotopic in U.

Star domains U are simply connected. Indeed, there exists a centre $a \in U$ such that $[a,z] \subset U$ for all $z \in U$. If $\gamma \colon [a,b] \to U$ is a closed curve, let $H(z,t) = (1-s)a + s\gamma(t) \in U$ for $(s,t) \in [0,1] \times [a,b]$. Then $H(s,t) \in U$, and H is a homotopy between γ and the constant curve $\gamma_0(t) = a$.

Theorem (Cauchy's theorem for simply connected domains)**.** If U is simply connected, then

$$\int_{\gamma} f(z)\,\mathrm{d}z = 0$$

for all holomorphic $f \colon U \to \mathbb{C}$, and every closed curve γ in U.

This is an immediate application of the above. The converse is also true, but is harder to prove.

Hence, U is simply connected if and only if $\int_\gamma f(z)\,\mathrm{d}z = 0$ for all holomorphic f and all closed γ in U. In particular, U is simply connected if and only if every closed curve in U is homologous to zero in U. Contrast this to the previous remark that if a curve is homologous to zero it is not necessarily null-homotopic.

4 Singularities

4.1 Motivation

Let U be open, and γ be a closed curve in U homologous to zero in U. Then, if $f : U \to \mathbb{C}$ is holomorphic, we have Cauchy's integral formula

$$\int_\gamma \underbrace{\frac{f(z)\,\mathrm{d}z}{z-a}}_{g(z)\mathrm{d}z} = 2\pi i \cdot I(\gamma; a) f(a)$$

for all $a \in U \setminus \operatorname{Im}\gamma$. This allows us to compute $\int_\gamma g(z)\,\mathrm{d}z$ for a holomorphic function $g : U \setminus \{a\} \to \mathbb{C}$ where γ does not pass through the point a, provided that g satisfies a particular condition: $(z-a)g(z)$ is the restriction to $U \setminus \{a\}$ of a holomorphic function $f : U \to \mathbb{C}$. We wish to drop this restriction and observe the consequences; that is, we wish to compute $\int_\gamma g(z)\,\mathrm{d}z$ for arbitrary holomorphic functions $g : U \setminus \{a\} \to \mathbb{C}$ for $a \in U$ and $a \notin \operatorname{Im}\gamma$. For example, consider $g(z) = e^{z^{-1}}$ for $U = \mathbb{C}$ and $a = 0$, $\gamma = \partial D(0,1)$. Note that $zg(z) = ze^{z^{-1}}$ is not continuous at $z = 0$, so it is certainly not holomorphic. This leads us to the study of singularities, and to eventually prove the residue theorem.

4.2 Removable singularities

Definition. Let $U \subseteq \mathbb{C}$ be open. If $a \in U$ and $f : U \setminus \{a\} \to \mathbb{C}$ is holomorphic, we say that f has an *isolated singularity* at a.

Definition. An isolated singularity a of f is a *removable singularity* if f can be defined at a such that the extended function is holomorphic on U.

Proposition. Let U be open, $a \in U$, and $f : U \setminus \{a\} \to \mathbb{C}$ be holomorphic. Then, the following are equivalent.
(i) f has a removable singularity at a;
(ii) $\lim_{z\to a} f(z)$ exists in $\mathbb{C}$;
(iii) there exists $D(a,\varepsilon) \subseteq U$ such that $|f(z)|$ is bounded in $D(a,\varepsilon) \setminus \{a\}$;
(iv) $\lim_{z\to a}(z-a)f(z) = 0$.

Proof. We can see that (i) implies (ii). If a is a removable singularity of f, then by definition there is a holomorphic function $g : U \to \mathbb{C}$ such that $f(z) = g(z)$ for all $z \in U \setminus \{a\}$. Then $\lim_{z\to a} f(z) = \lim_{z\to a} g(z) = g(a) \in \mathbb{C}$. Similarly, (ii) implies (iii) and (iii) implies (iv) are clear.

It suffices to check (iv) implies (i). Consider the function

$$h(z) = \begin{cases} (z-a)^2 f(z) & \text{if } z \neq a \\ 0 & \text{if } z = a \end{cases}$$

We have

$$\lim_{z \to a} \frac{h(z) - h(a)}{z - a} = \lim_{z \to a} (z - a) f(z) = 0$$

Hence h is differentiable at a with $h'(a) = 0$. Since h is differentiable in $U \setminus \{a\}$, we must have that h is holomorphic in U. Since $h(a) = h'(a) = 0$, we can find $r > 0$ and a holomorphic $g \colon D(a, r) \to \mathbb{C}$ such that $h(z) = (z - a)^2 g(z)$ for $z \in D(a, r)$. Comparing this to the definition of h, we have that $f(z) = g(z)$ for $D(a, r) \setminus \{a\}$. By defining $f(a) = g(a)$, we have that f is differentiable at a with $f'(a) = g'(a)$. So a is a removable singularity of f. $\square$

Example. Consider $f(z) = \frac{e^z - 1}{z}$. Certainly f is holomorphic on $\mathbb{C} \setminus \{0\}$, and $\lim_{z \to 0} z f(z) = 0$. So $z = 0$ is a removable singularity. We can also see directly by the Taylor series of e^z at $z = 0$ that $f(z) = \sum_{k=1}^{\infty} \frac{z^{k-1}}{k!}$ for $z \neq 0$, and the series on the right hand side defines an entire function.

Remark. If $u \colon D(0, 1) \setminus \{0\} \to \mathbb{R}$ is a C^2 harmonic function, when can we say that $z = 0$ is a removable singularity, i.e. that u extends to $z = 0$ as a harmonic function? We can relate this to the study of holomorphic functions. However, unlike with previous cases, the analogy is more subtle in this case. We cannot necessarily construct a harmonic conjugate v such that $u + iv$ is holomorphic in $D(0, 1) \setminus \{0\}$, because U is not simply connected.

There is a similar result, however. If $\lim_{z \to 0} u(z)$ exists, then the extended function is in fact C^2 and harmonic. More generally, if u is bounded near $z = 0$, there exists a harmonic extension. We can also consider the case $\lim_{z \to 0} |z||u(z)| = 0$; this is explored on the example sheets.

4.3 Poles

Note, if a holomorphic function f has a non-removable singularity, f is not bounded in $D(a, r) \setminus \{a\}$ for any $r > 0$.

Definition. If $a \in U$ is an isolated singularity of f, then a is a *pole* of f if

$$\lim_{z \to a} |f(z)| = \infty$$

Example. $f(z) = (z - a)^{-k}$ for $k \in \mathbb{N}$ has a pole at a.

Definition. If $a \in U$ is an isolated singularity of f that is not removable or a pole, it is an *essential singularity*.

Remark. An equivalent characterisation for a to be an essential singularity is that the limit $\lim_{z \to a} |f(z)|$ does not exist. This follows from the previous proposition and the definition of a pole.

Example. $f(z) = e^{\frac{1}{z}}$ has $|f(iy)| = 1$ for all $y \in \mathbb{R} \setminus \{0\}$ and $\lim_{x \to 0^+} f(x) = \infty$. So $z = 0$ is an essential singularity of f.

Proposition. Let $f \colon U \setminus \{a\} \to \mathbb{C}$ be holomorphic. The following are equivalent.

(i) f has a pole at a;

(ii) there exists $\varepsilon > 0$ and a holomorphic function $h \colon D(a, \varepsilon) \to \mathbb{C}$ with $h(a) = 0$ and

$h(z) \neq 0$ for all $z \neq a$ such that $f(z) = \frac{1}{h(z)}$ for $z \in D(a, \varepsilon) \setminus \{a\}$;

(iii) there exists a unique integer $k \geq 1$ and a unique holomorphic function $g : U \to \mathbb{C}$ with $g(a) \neq 0$ such that $f(z) = (z-a)^{-k} g(z)$ for $z \in U \setminus \{a\}$.

Remark. Since (i) implies (iii), there exists no holomorphic function on a punctured disc $f : D(a,R)\setminus \{a\} \to \mathbb{C}$ such that $|f(z)| \to \infty$ as $z \to a$ at the rate of a negative non-integer power of $|z-a|$, i.e. with $c|z-a|^{-s} \leq |f(z)| \leq C|z-a|^{-s}$ for some constants $s \in (0,\infty) \setminus \mathbb{N}$, $c > 0$, $C > 0$, and all $z \in D(a,R) \setminus \{a\}$.

Proof. We show (i) implies (ii). Since $\lim_{z\to a} |f(z)| = \infty$, there exists $\varepsilon > 0$ such that $|f(z)| \geq 1$ for all $0 < |z-a| < \varepsilon$. Hence $\frac{1}{f(z)}$ is holomorphic and bounded in $D(a,\varepsilon) \setminus \{a\}$. By the above proposition, $\frac{1}{f}$ has a removable singularity at a, so there exists a holomorphic function $h : D(a,\varepsilon) \to \mathbb{C}$ such that $\frac{1}{f} = h$, or equivalently, $f = \frac{1}{h}$, for $z \in D(a,\varepsilon)\setminus\{a\}$. Since $|f(z)| \to \infty$ as $z \to a$, we have that $h(a) = 0$.

Now we show (ii) implies (iii). Let ε and h be as in the definition of (ii). By Taylor series, there exists $k \geq 1$ and a holomorphic function $h_1 : D(a,\varepsilon) \to \mathbb{C}$ with $h_1(z) \neq 0$ for all $z \in D(a,\varepsilon)$ such that $h(z) = (z-a)^k h_1(z)$. If $g_1 = \frac{1}{h_1}$, then g_1 is holomorphic in $D(a,\varepsilon)$, $g_1 \neq 0$ in $D(a,\varepsilon)$, and $f(z) = (z-a)^{-k} g_1(z)$ in $D(a,\varepsilon) \setminus \{a\}$.

We can now define $g : U \to \mathbb{C}$ by $g(z) = g_1(z)$ for $z \in D(a,\varepsilon)$, and $g(z) = (z-a)^k f(z)$ for $z \in U \setminus \{a\}$. Since $f(z) = (z-a)^{-k} g_1(z)$, the definitions agree on $D(a,\varepsilon)\setminus\{a\}$, so g is well-defined and holomorphic in U, and $g(a) = g_1(a) \neq 0$. This proves the existence of an integer $k \geq 1$ and a holomorphic $g : U \to \mathbb{C}$ with $g(a) \neq 0$ such that $f(z) = (z-a)^{-k} g(z)$ for all $z \in U \setminus \{a\}$.

To prove uniqueness of k and g, suppose there exists $\tilde{k} \geq 1$ and a holomorphic $\tilde{g} : U \to \mathbb{C}$ with $\tilde{g}(a) \neq 0$ such that $f(z) = (z-a)^{-\tilde{k}} \tilde{g}(z)$ for all $z \in U \setminus \{a\}$. Then we have $g(z) = (z-a)^{k-\tilde{k}} \tilde{g}(z)$ for $z \in U \setminus \{a\}$. Since $g, \tilde{g}$ are holomorphic with $g(a) \neq 0$ and $\tilde{g}(a) \neq 0$, this can only be true if $k = \tilde{k}$, and hence $g = \tilde{g}$ on $U \setminus \{a\}$, and then at a by continuity.

It is clear that (iii) implies (i). □

Definition. If f has a pole at $z = a$, then the unique positive integer k given by the above proposition is the *order* of the pole at a. If $k = 1$, we say that f has a *simple pole* at a.
Let U be open and $S\setminus U$ be a discrete subset of U, so all points of S are isolated. If $f : U\setminus S \to \mathbb{C}$ is holomorphic and each $a \in S$ is either a removable singularity or a pole of f, then f is a *meromorphic* function on U. In particular, if $S = \varnothing$, f is holomorphic.

Remark. If $f : U \setminus \{a\} \to \mathbb{C}$ is holomorphic and the singularity $z = a$ is a pole of f, we can regard f as a continuous mapping onto the Riemann sphere $f : U \to \mathbb{C} \cup \{\infty\}$, by setting $f(a) = \infty$. Here, f is holomorphic on U. Holomorphicity of the extended map near the pole a follows from the fact that in a punctured disc about a, $\frac{1}{f}$ has the form $\frac{(z-a)^k}{g(z)}$ for some holomorphic g with $g(z) \neq 0$ near a; and the fact that any function h defined in a neighbourhood of ∞ in the Riemann sphere is holomorphic, by definition, if the function $\widetilde{h}(z) = h\left(\frac{1}{z}\right)$ if $z \neq 0$, $\widetilde{h}(0) = h(\infty)$ is holomorphic near zero. Hence $h \circ f = \widetilde{h} \circ \left(\frac{1}{f}\right)$ is holomorphic near a for all holomorphic h in a neighbourhood of ∞ in the Riemann sphere.

Hence, any meromorphic function $f: U \setminus S \to \mathbb{C}$ can be viewed as a holomorphic function $U \to \mathbb{C} \cup \{\infty\}$. Geometrically, therefore, poles are not 'real' singularities, and the only true isolated singularities are the essential singularities. This is explored further in Part II Riemann Surfaces.

4.4 Essential singularities

Remark. Suppose $z = a$ is an essential singularity of a holomorphic $f: U \setminus \{a\} \to \mathbb{C}$. Then there exists a sequence of points $a_n \in U \setminus \{a\}$, $a_n \to a$, such that $f(a_n) \to \infty$. This is because a is not removable. There is also another sequence of points $b_n \in U \setminus \{a\}$, $b_n \to a$ such that $(f(b_n))$ is bounded. This is because a is not a pole. We can generalise this further.

Theorem (Casorati–Weierstrass theorem). If $f: U \setminus \{a\} \to \mathbb{C}$ is holomorphic and $a \in U$ is an essential singularity of f, then for any $\varepsilon > 0$, the set $f(D(a, \varepsilon) \setminus \{a\})$ is dense in $\mathbb{C}$.

The proof is an exercise on the second example sheet.

Theorem (Picard's theorem). If $f: U \setminus \{a\} \to \mathbb{C}$ is holomorphic and $a \in U$ is an essential singularity of f, then there exists $w \in \mathbb{C}$ such that for any $\varepsilon > 0$, $\mathbb{C} \setminus \{w\} \subseteq f(D(a, \varepsilon) \setminus \{a\})$. In other words, in any neighbourhood $D(a, \varepsilon) \setminus \{a\}$, f attains all complex numbers except possibly one.

The proof is omitted.

4.5 Laurent series

If $z = a$ is a removable singularity of f, then for some $R > 0$, f is given by a power series $\sum_{n=0}^{\infty} c_n (z - a)^n$, which is the Taylor series of the holomorphic extension of f to $D(a, R)$, for all $z \in D(a, R) \setminus \{a\}$. If a is a pole of some order $k \geq 1$, then for some $R > 0$ we have $f(z) = (z - a)^{-k} g(z)$ for some holomorphic $g: D(a, R) \to \mathbb{C}$ and all $z \in D(a, R) \setminus \{a\}$, so using the Taylor seires of g, we find a series of the form $f(z) = \sum_{n=-k}^{\infty} c_n (z - a)^n$, for $z \in D(a, R) \setminus \{a\}$. When a is an essential singularity, we can still obtain an analogous series expansion with infinitely many terms with negative powers. More generally, we have the following.

Theorem (Laurent expansion). Let f be holomorphic on an annulus

$$A = \{z \in \mathbb{C} : r < |z - a| < R\}$$

for $0 \leq r < R \leq \infty$. Then:

(i) f has a unique convergent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n \equiv \sum_{n=1}^{\infty} c_{-n} (z - a)^{-n} + \sum_{n=0}^{\infty} c_n (z - a)^n$$

where the c_n are constants;

(ii) for any $\rho \in (r, R)$, the coefficient c_n is given by

$$c_n = \frac{1}{2\pi i} \int_{\partial D(a, \rho)} \frac{f(z) \, \mathrm{d}z}{(z - a)^{n+1}}$$

(iii) if $r < \rho' \leq \rho < R$, then the two series in (i) separately converge uniformly on the set

$$\{z \in \mathbb{C} : \rho' \leq |z - a| \leq \rho\}$$

Remark. If f is the restriction of A of a holomorphic function g on the full disc $D(a, R)$, then by the formula in part (ii), we have for any negative $n = -m$, $m \geq 1$, the coefficient c_{-m} is zero by Cauchy's theorem. In this case, the Laurent series of f is the Taylor series of g restricted to A. The new content of the theorem is simply when f has no holomorphic extension to $D(a, R)$.

Proof. Let $w \in A$ and consider the function

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} & \text{if } z \neq w \\ f'(w) & \text{if } z = w \end{cases}$$

This g is continuous in A and holomorphic in $A \setminus \{w\}$. Hence, this is holomorphic in A since this is a removable singularity. Let ρ_1, ρ_2 such that $r < \rho_1 < |w - a| < \rho_2 < R$. The two positively oriented curves $\partial D(a, \rho_1)$ and $\partial D(a, \rho_2)$ are homotopic in A. Hence,

$$\int_{\partial D(a,\rho_1)} g(z)\,\mathrm{d}z = \int_{\partial D(a,\rho_2)} g(z)\,\mathrm{d}z$$

Substituting for g,

$$\int_{\partial D(a,\rho_1)} \frac{f(z)\,\mathrm{d}z}{z-w} - 2\pi i \cdot I(\partial D(a, \rho_1); w) f(w) = \int_{\partial D(a,\rho_2)} \frac{f(z)\,\mathrm{d}z}{z-w} - 2\pi i \cdot I(\partial D(a, \rho_2); w) f(w)$$

We have

$$I(\partial D(a, \rho_1); w) = 0; \quad I(\partial D(a, \rho_2); w) = I(\partial D(a, \rho_2); a) = 1$$

Hence,

$$f(w) = \frac{1}{2\pi i} \int_{\partial D(a,\rho_2)} \frac{f(z)\,\mathrm{d}z}{z-w} - \frac{1}{2\pi i} \int_{\partial D(a,\rho_1)} \frac{f(z)\,\mathrm{d}z}{z-w}$$

This is an analogue of Cauchy's integral formula for annular domains. We can now proceed as before when proving the Taylor series expansion for holomorphic functions.

For the first integral, consider the expansion

$$\frac{1}{z-w} = \frac{1}{z-a-(w-a)} = \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}}$$

This series converges uniformly over $z \in \partial D(a, \rho_2)$, since $\left|\frac{w-a}{z-a}\right| < 1$. For the second integral, consider

$$\frac{1}{z-w} = \frac{1}{z-a-(w-a)} = -\frac{1}{(w-a)\left(1 - \frac{z-a}{w-a}\right)} = -\sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}}$$

Likewise, this series converges uniformly over $z \in \partial D(a, \rho_1)$, since $\left|\frac{z-a}{w-a}\right| < 1$ in this disc. Substituting these into the representation formula, we can switch integration and summation due to uniform convergence. This gives

$$f(w) = \sum_{n=0}^{\infty} c_n (w-a)^n + \sum_{n=1}^{\infty} c_{-n} (w-a)^{-n}$$

where
$$c_n = \frac{1}{2\pi i}\int_{\partial D(a,\rho_2)} \frac{f(z)\,\mathrm{d}z}{(z-a)^{n+1}}$$
for $n \geq 0$, and
$$c_n = \frac{1}{2\pi i}\int_{\partial D(a,\rho_1)} \frac{f(z)\,\mathrm{d}z}{(z-a)^{n+1}}$$
for $n \leq -1$. Since $\partial D(a,\rho_1)$ and $\partial D(a,\rho_2)$ are homotopic in A to $\partial D(a,\rho)$ for any $\rho \in (r,R)$, we have that
$$c_n = \frac{1}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(z)\,\mathrm{d}z}{z-a}$$
for any $\rho \in (r,R)$ and $n \in \mathbb{Z}$, so (i) and the formula (ii) both hold.

To show (iii) and uniqueness, suppose there exist constants c_n such that, for all $z \in A$, we have
$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n \tag{$*$}$$
Let $r < \rho' \leq \rho < R$. Then the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges for $z \in A$, so it has radius of convergence at least R, and converges uniformly for $|z-a| \leq \rho$. Further, the series $\sum_{n=1}^{\infty} c_{-n}(z-a)^{-n}$ converges on A. Let $\zeta = (z-a)^{-1}$. Then the power series $\sum_{n=1}^{\infty} c_{-n}\zeta^n$ converges for $\frac{1}{R} < |\zeta| < \frac{1}{r}$ so it has radius of convergence at least $\frac{1}{r}$ and converges uniformly for $|\zeta| \leq \frac{1}{\rho'}$. Thus, the series $\sum_{n=1}^{\infty} c_{-n}(z-a)^{-n}$ converges uniformly for $|z-a| \geq \rho'$. Hence $(*)$ converges uniformly in $\rho' \leq |z-a| \leq \rho$. Hence, for any $m \in \mathbb{Z}$, we have
$$\int_{\partial D(a,\rho)} \frac{f(z)\,\mathrm{d}z}{(z-a)^{m+1}} = \sum_{n=-\infty}^{\infty} c_n \int_{\partial D(a,\rho)} (z-a)^{n-m-1}\,\mathrm{d}z$$
By the fundamental theorem of calculus, the only nonzero integral on the right hand side occurs when $n-m-1=-1$, which occurs for $n=m$ only. This integral gives
$$c_m = \frac{1}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(z)\,\mathrm{d}z}{(z-a)^{m+1}}$$
for all $\rho \in (r,R)$. This formula also implies the uniqueness of the c_n for which the series expansion is valid. □

Remark. The above proof shows that if $f : A \equiv D(a,R)\setminus\overline{D(a,r)} \to \mathbb{C}$ is holomorphic, then there is a holomorphic function $f_1 : D(a,R) \to \mathbb{C}$ and a holomorphic function $f_2 : \mathbb{C}\setminus\overline{D(a,r)} \to \mathbb{C}$ such that $f = f_1 + f_2$ on A. This decomposition is not unique, since we can take $f_1 \mapsto f_1 + g$ and $f_2 \mapsto f_2 - g$ for an entire function g. However, if we also require $f_2(z) \to 0$ as $z \to \infty$, the decomposition into two series given in (ii) above is unique.

4.6 Coefficients of Laurent series

Let $f : D(a,R)\setminus\{a\} \to \mathbb{C}$ be holomorphic, so $z = a$ is an isolated singularity of f. Then, by the Laurent series with $r = 0$, we have a unique set of complex numbers c_n such that
$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

Then,

(i) If $c_n = 0$ for all $n < 0$, we have $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n \equiv g(z)$ on $D(a,R) \setminus \{a\}$. Since g is holomorphic on $D(a,R)$, $z = a$ is a removable singularity.

(ii) If $c_{-k} \neq 0$ for some $k \geq 1$ and $c_{-n} = 0$ for all $n \geq k+1$, we have

$$f(z) = \frac{c_{-k}}{(z-a)^k} + \frac{c_{-k+1}}{(z-a)^{k+1}} + \dots + \frac{c_{-1}}{z-a} + \sum_{n=0}^{\infty} c_n(z-a)^n$$

Hence, $f(z) = (z-a)^{-k} g(z)$ for a function g which is holomorphic on $D(a,R)$, and where $g(a) = c_{-k} \neq 0$. Equivalently, $z = a$ is a pole of order k.

(iii) If $c_n \neq 0$ for infinitely many $n < 0$, $z = a$ is an essential singularity. This holds since the above two parts were all bidirectional implications.

4.7 Residues

Definition. Let $f : D(a,R) \setminus \{a\} \to \mathbb{C}$ be holomorphic. The coefficient c_{-1} of the Laurent series of f in $D(a,R) \setminus \{a\}$ is called the *residue of f at a*, denoted $\operatorname{Res}_f(a)$. The series

$$f_P = \sum_{n=1}^{\infty} c_{-n}(z-a)^{-n}$$

is known as the *principal part of f at a*.

We know that f_P is holomorphic on $\mathbb{C} \setminus \{a\}$, with the series defining f_P converging uniformly on compact subsets of $\mathbb{C} \setminus \{a\}$. By the Laurent series, $f = f_P + h$ on $D(a,R) \setminus \{a\}$, where h is holomorphic on $D(a,R)$. Let γ be a closed curve in $D(a,R)$ with $a \notin \operatorname{Im}\gamma$. Then $\int_\gamma h(z)\,\mathrm{d}z = 0$ by Cauchy's theorem, and hence $\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma f_P(z)\,\mathrm{d}z = 2\pi i \cdot I(\gamma; a) \operatorname{Res}_f(a)$, where the last inequality holds by uniform convergence of the series for f_P and the fundamental theorem of calculus. This reasoning can be extended to the case of more then one isolated singularity.

Theorem (residue theorem)**.** Let U be an open set, $\{a_1, \dots, a_k\} \subset U$ be finite, and $f : U \setminus \{a_1, \dots, a_k\} \to \mathbb{C}$ be holomorphic. If γ is a closed curve in U homologous to zero in U, and if $a_j \notin \operatorname{Im}\gamma$ for each j, then

$$\frac{1}{2\pi i} \int_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^{k} I(\gamma; a_j) \operatorname{Res}_f(a_j)$$

This is a generalisation of Cauchy's integral formula.

Proof. Let $f_P^{(j)} = \sum_{n=1}^{\infty} c_{-n}^{(j)}(z-a_j)^{-n}$ be the principal part of f at a_j. Then $f_P^{(j)}$ is holomorphic in $\mathbb{C} \setminus \{a_j\}$, and hence is holomorphic in $\mathbb{C} \setminus \{a_1, \dots, a_k\}$. Let

$$h \equiv f - \left(f_P^{(1)} + \dots + f_P^{(k)}\right)$$

This h is holomorphic in $U \setminus \{a_1, \dots, a_k\}$. Let j be fixed. Then $f - f_P^{(j)}$ has a removable singularity at $z = a_j$. For all $\ell \neq j$, $f_P^{(\ell)}$ is holomorphic at a_j. Hence h has a removable singularity at a_j. This is true for all j, so h extends to all of U as a holomorphic function. By Cauchy's theorem, $\int_\gamma h(z)\,\mathrm{d}z = 0$. Hence

$$\frac{1}{2\pi i}\int_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^{k} \frac{1}{2\pi i}\int_\gamma f_P^{(j)}(z)\,\mathrm{d}z$$

By termwise integration of the series for $f_P^{(j)}$, which converges uniformly on compact subsets of $\mathbb{C} \setminus \{a_j\}$, we have

$$\frac{1}{2\pi i}\int_\gamma f_P^{(j)}(z)\,\mathrm{d}z = I(\gamma; a_j)\,\mathrm{Res}_f(a_j)$$

as required. □

There are simple ways to calculate residues if we know information about the singularity in question.

(i) If f has a simple pole at $z = a$, then

$$\mathrm{Res}_f(a) = \lim_{z\to a}(z-a)f(z)$$

Indeed, near a, we have $f(z) = (z-a)^{-1}g(z)$ where g is holomorphic and $g(a) \neq 0$. Hence, by the Taylor expansion of g, we have that $\mathrm{Res}_f(a) = g(a)$.

(ii) If f has a pole of order k at a, then near a we have that $f(z) = (z-a)^{-k}g(z)$ where g is holomorphic and $g(a) \neq 0$. In this case, the residue $\mathrm{Res}_f(a)$ is the coefficient of the $(z-a)^{k-1}$ term of the Taylor series of g at a, which is

$$\mathrm{Res}_f(a) = \frac{g^{(k-1)}(a)}{(k-1)!}$$

(iii) If $f = \frac{g}{h}$ where g and h are holomorphic at $z = a$, such that $g(a) \neq 0$ and h has a simple zero at $z = a$, then from (i) we have

$$\mathrm{Res}_f(a) = \lim_{z\to a}\frac{(z-a)g(z)}{h(z)} = \lim_{z\to a}\frac{g(z)}{\frac{h(z)-h(a)}{z-a}} = \frac{g(a)}{h'(a)}$$

Example. For $0 < \alpha < 1$, we will show that

$$\int_0^\infty \frac{x^{-\alpha}}{1+x}\,\mathrm{d}x = \frac{\pi}{\sin \pi\alpha}$$

Let $g(z) = z^{-\alpha}$ be the branch of $z^{-\alpha}$ defined by $g(z) = e^{-\alpha\ell(z)}$, where $\ell(z)$ is the holomorphic branch of logarithm on $U = \mathbb{C} \setminus \{x \in \mathbb{R} : x \geq 0\}$. given by $\ell(z) = \log|z| + i\arg z$ where $\arg(z)$ takes values in $(0, 2\pi)$. Let $f(z) = \frac{g(z)}{1+z}$. Then

$$f(z) = \frac{|z|^{-\alpha}e^{-i\alpha\arg z}}{1+z}$$

and f is holomorphic in $U \setminus \{-1\}$ where $z = -1$ is a simple pole with $\mathrm{Res}_f(-1) = \lim_{z\to -1}(z+1)f(z) = e^{-i\pi\alpha}$.

Let ε, R be such that $0 < \varepsilon < 1 < R$ and $\theta > 0$ be small. Let γ be the positively-oriented 'keyhole countour' determined by the two circular arcs $\gamma_R : [\theta, 2\pi - \theta] \to U$ and the two line segments $\gamma_1, \gamma_2 : [\varepsilon, R] \to U$ given by

$$\gamma_R(t) = Re^{it}; \quad \gamma_\varepsilon(t) = \varepsilon e^{i(2\pi - t)}; \quad \gamma_1(t) = te^{i\theta}; \quad \gamma_2(t) = te^{i(2\pi-\theta)}$$

The domain U is star shaped and hence simply connected, and so γ is homologous to zero in U. Directly from the definitions of γ and the winding number, we can show that $I(\gamma; -1) = 1$.

By the residue theorem, we find $\int_\gamma f(z)\,\mathrm{d}z = 2\pi i e^{-i\pi\alpha}$. Now,

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_\varepsilon^R f(te^{i\theta})e^{i\theta}\,\mathrm{d}t = \int_\varepsilon^R \frac{t^{-\alpha}e^{i(1-\alpha)\theta}}{1+te^{i\theta}}\,\mathrm{d}t$$

and

$$\int_{\gamma_2} f(z)\,\mathrm{d}z = \int_\varepsilon^R f(te^{i(2\pi-\theta)})e^{i(2\pi-\theta)}\,\mathrm{d}t = \int_\varepsilon^R \frac{t^{-\alpha}e^{i(1-\alpha)(2\pi-\theta)}}{1+te^{i(2\pi-\theta)}}\,\mathrm{d}t$$

As $\theta \to 0^+$, we can show that the integrands converge uniformly on $[\varepsilon, R]$ to $\frac{t^{-\alpha}}{1+t}$ and $\frac{e^{-2i\pi\alpha}t^{-\alpha}}{1+t}$ respectively. Hence,

$$\lim_{\theta\to 0^+}\left[\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{(-\gamma_2)} f(z)\,\mathrm{d}z\right] = \left(1 - e^{-2i\pi\alpha}\right)\int_\varepsilon^R \frac{t^{-\alpha}}{1+t}\,\mathrm{d}t$$

For all $z \in \operatorname{Im}\gamma_R$, we have $|f(z)| \le \frac{R^{-\alpha}}{R-1}$; and for all $z \in \operatorname{Im}\gamma_\varepsilon$, we have $|f(z)| \le \frac{\varepsilon^{-\alpha}}{1-\varepsilon}$. Hence,

$$\left|\int_{\gamma_R} f(z)\,\mathrm{d}z + \int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z\right| \le \frac{2\pi R^{1-\alpha}}{R-1} + \frac{2\pi\varepsilon^{1-\alpha}}{1-\varepsilon}$$

Note that the right hand side is independent of θ, even though γ_R and γ_ε depend on θ. Since

$$\int_\gamma f(z)\,\mathrm{d}z - \left(\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{(-\gamma_2)} f(z)\,\mathrm{d}z\right) = \int_{\gamma_R} f(z)\,\mathrm{d}z + \int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z$$

we then have that

$$\left|2\pi i e^{-i\pi\alpha} - \left(\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{(-\gamma_2)} f(z)\,\mathrm{d}z\right)\right| \le \frac{2\pi R^{1-\alpha}}{R-1} + \frac{2\pi\varepsilon^{1-\alpha}}{1-\varepsilon}$$

First letting $\theta \to 0^+$ in this, and then letting $\varepsilon \to 0^+$ and $R \to \infty$, we conclude

$$\left(1 - e^{-2\pi i\alpha}\right)\int_0^\infty \frac{t^{-\alpha}}{1+t}\,\mathrm{d}t = 2\pi i e^{-i\pi\alpha}$$

or,

$$\int_0^\infty \frac{t^{-\alpha}}{1+t}\,\mathrm{d}t = \frac{\pi}{\sin\pi\alpha}$$

4.8 Jordan's lemma

Lemma. Let f be a continuous complex-valued function on the semicircle $C_R^+ = \operatorname{Im}\gamma_R^+$ in the upper half-plane, where $R > 0$ and $\gamma_R^+(t) = Re^{it}$ for $0 \le t \le \pi$. Then, for $\alpha > 0$,

$$\left|\int_{\gamma_R^+} f(z)e^{i\alpha z}\,\mathrm{d}z\right| \le \frac{\pi}{\alpha}\sup_{z\in C_R^+}|f(z)|$$

In particular, if f is continuous in $H^+ \setminus D(0, R_0)$ for $R_0 > 0$ where $H^+ = \{z: \operatorname{Im} z \ge 0\}$ and if $\sup_{z\in C_R^+}|f(z)| \to 0$ as $R \to \infty$, then for each $\alpha > 0$, we have

$$\int_{\gamma_R^+} f(z)e^{i\alpha z}\,\mathrm{d}z \to 0$$

as $R \to \infty$.

A similar statement holds for $\alpha < 0$ and the semicircle $C_R^- = \operatorname{Im}\gamma_R^-$ in the lower half-plane where $\gamma_R^-(t) = -Re^{it}$ for $R > 0$ and $0 \le t \le \pi$.

Proof. Let $M_R = \sup_{z\in C_R^+}|f(z)|$. Then,

$$\begin{aligned}
\left|\int_{\gamma_R^+} f(z)e^{i\alpha z}\,\mathrm{d}z\right| &= \left|\int_0^{\pi} f(Re^{it})e^{-\alpha R\sin t + i\alpha R\cos t} iRe^{it}\,\mathrm{d}t\right| \\
&\le RM_R\int_0^{\pi} e^{-\alpha R\sin t}\,\mathrm{d}t \\
&= RM_R\left(\int_0^{\frac{\pi}{2}} e^{-\alpha R\sin t}\,\mathrm{d}t + \int_{\frac{\pi}{2}}^{\pi} e^{-\alpha R\sin t}\,\mathrm{d}t\right) \\
&= 2RM_R\int_0^{\frac{\pi}{2}} e^{-\alpha R\sin t}\,\mathrm{d}t \\
&\le 2RM_R\int_0^{\frac{\pi}{2}} e^{\frac{-2\alpha R t}{\pi}}\,\mathrm{d}t \\
&= \frac{\pi M_R}{\alpha}\left(1 - e^{-\alpha R}\right) \le \frac{\pi M_R}{\alpha}
\end{aligned}$$

where we have used the fact that for $t \in (0, \frac{\pi}{2}]$, $\varphi(t) \equiv \frac{\sin t}{t} \ge \frac{2}{\pi}$ since $\varphi(\frac{\pi}{2}) = \frac{2}{\pi}$ and $\varphi'(t) \le 0$ on $[0, \frac{\pi}{2}]$. □

Lemma (integrals on small circular arcs)**.** Let f be holomorphic in $D(a, R)\backslash\{a\}$ with a simple pole at $z = a$. Let $\gamma_\varepsilon : [\alpha, \beta] \to \mathbb{C}$ be the circular arc $\gamma_\varepsilon(t) = a + \varepsilon e^{it}$. Then

$$\lim_{\varepsilon\to 0^+}\int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z = (\beta - \alpha)i\operatorname{Res}_f(a)$$

Proof. Let $f(z) = \frac{c}{z-a} + g(z)$ where g is holomorphic in $D(a, R)$ and $c = \mathrm{Res}_f(a)$. Then

$$\left|\int_{\gamma_\varepsilon} g(z)\,\mathrm{d}z\right| = \left|\int_\alpha^\beta g(a + \varepsilon e^{it})\varepsilon i e^{it}\right| \le \varepsilon(\beta - \alpha) \sup_{t\in[\alpha,\beta]} |g(a + \varepsilon e^{it})| \to 0$$

as $\varepsilon \to 0^+$. By direct calculation,

$$\int_{\gamma_\varepsilon} \frac{c}{z-a}\,\mathrm{d}z = (\beta - \alpha)i\,\mathrm{Res}_f(a)$$

Hence the claim follows. $\square$

Example. Consider $\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x$. Let $f(z) = \frac{e^{iz}}{z}$. Consider the integral $\int_\gamma f(z)\,\mathrm{d}z$ over the curve $\gamma = \gamma_R + \gamma_1 + \gamma_\varepsilon + \gamma_2$ where

(i) $\gamma_R(t) = Re^{it}$ for $0 \le t \le \pi$;

(ii) $\gamma_1(t) = t$ for $-R \le t \le -\varepsilon$;

(iii) $\gamma_\varepsilon(t) = \varepsilon e^{-it}$ for $-\pi \le t \le 0$;

(iv) $\gamma_2(t) = t$ for $\varepsilon \le t \le R$.

By Jordan's lemma, $\int_{\gamma_R} f(z)\,\mathrm{d}z \to 0$ as $R \to \infty$. f has a simple pole at $z = 0$ with $\mathrm{Res}_f(0) = \lim_{z\to 0} zf(z) = 1$. By the above lemma, $\int_{-\gamma_\varepsilon} f(z)\,\mathrm{d}z \to \pi i$ as $\varepsilon \to 0^+$.

Since f is holomorphic in $U = \mathbb{C} \setminus \{0\}$ and γ is homologous to zero in U, Cauchy's theorem gives that

$$\int_\gamma f(z)\,\mathrm{d}z = 0 \implies \int_{\gamma_R} f(z)\,\mathrm{d}z + \int_{-R}^{-\varepsilon} \frac{e^{it}}{t}\,\mathrm{d}t + \int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z + \int_\varepsilon^R \frac{e^{it}}{t}\,\mathrm{d}t = 0$$

Combining the two integrals on the real axis under a change of variables,

$$\int_\varepsilon^R \frac{e^{it} - e^{-it}}{t}\,\mathrm{d}t + \int_{\gamma_R} f(z)\,\mathrm{d}z + \int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z = 0$$

Letting $R \to \infty$ and $\varepsilon \to 0^+$, we have

$$\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \frac{\pi}{2}$$

Example. We prove that $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$. Consider the function

$$f(z) = \frac{\pi\cot(\pi z)}{z^2} = \frac{\pi\cos(\pi z)}{z^2\sin(\pi z)}$$

This is holomorphic in $\mathbb{C}$ except for simple poles at each point in $\mathbb{Z} \setminus \{0\}$, and an order 3 pole at zero. Near $n \in \mathbb{Z} \setminus \{0\}$, we have $f(z) = \frac{g(z)}{h(z)}$ where $g(n) \neq 0$ and h has a simple zero at n, and so

$$\mathrm{Res}_f(n) = \frac{g(n)}{h'(n)} = \frac{1}{n^2}$$

To compute the residue at zero, consider

$$\cot z = \frac{\cos z}{\sin z} = \left(1 - \frac{z^2}{2} + O(z^4)\right)\left(z - \frac{z^3}{6} + O(z^5)\right)^{-1} = \frac{1}{z} - \frac{z}{3} + O(z^2)$$

Hence,

$$\frac{\pi\cot(\pi z)}{z^2} = \frac{1}{z^3} - \frac{\pi^2}{3z} + \dots$$

This shows that $\mathrm{Res}_f(0) = -\frac{\pi^2}{3}$. For $N \in \mathbb{N}$, let γ_N be the positively oriented boundary of the square defined by the lines $x = \pm(N + \frac{1}{2})$ and $y = \pm(N + \frac{1}{2})$. By the residue theorem,

$$\int_{\gamma_N} f(z)\,\mathrm{d}z = 2\pi i\left[2\left(\sum_{n=1}^{N} \frac{1}{n^2}\right) - \frac{\pi^2}{3}\right] \tag{$*$}$$

Since $\mathrm{length}(\gamma_N) = 4(2N+1)$, we have

$$\begin{aligned}\left|\int_{\gamma_N} f(z)\,\mathrm{d}z\right| &\le \sup_{\gamma_N}\left|\frac{\pi\cot(\pi z)}{z^2}\right| \cdot 4(2N+1) \\ &\le \sup_{\gamma_N}|\cot(\pi z)| \cdot \frac{4(2N+1)\pi}{(N+\frac{1}{2})^2} \\ &= \frac{16\pi}{2N+1} \cdot \sup_{\gamma_N}|\cot(\pi z)|\end{aligned}$$

On γ_N, it is possible to show that $\cot(\pi z)$ is bounded independently of N. Hence,

$$\int_{\gamma_N} f(z)\,\mathrm{d}z \to 0$$

as $N \to \infty$. Letting $N \to \infty$ in $(*)$, we find

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

5 The argument principle, local degree, and Rouché's theorem

5.1 The argument principle

Proposition. If f has a zero (or pole) of order $k \ge 1$ at $z = a$, then $\frac{f'}{f}$ has a simple pole at $z = a$ with residue k (or $-k$, respectively).

Proof. If $z = a$ is a zero of order k, there is a disc $D(a,r)$ such that $f(z) = (z-a)^k g(z)$ for $z \in D(a,r)$ where $g : D(a,r) \to \mathbb{C}$ is holomorphic with $g(z) \neq 0$ for all $z \in D(a,r)$. Hence,

$$f'(z) = k(z-a)^{k-1}g(z) + (z-a)^k g'(z)$$

and

$$\frac{f'(z)}{f(z)} = \frac{k}{z-a} + \frac{g'(z)}{g(z)}$$

for all $z \in D(a,r) \setminus \{a\}$. Since $\frac{g'}{g}$ is holomorphic in $D(a,R)$, the claim follows. A similar argument holds for poles. □

Definition. The order of a zero or pole a of a holomorphic function f is denoted $\operatorname{ord}_f(a)$.

Theorem (the argument principle)**.** Let f be a meromorphic function on a domain U with finitely many zeroes $a_1, \dots, a_k$ and finitely many poles $b_1, \dots, b_\ell$. If γ is a closed curve in U homologous to zero in U, and if $a_i, b_j \notin \operatorname{Im}\gamma$ for all i, j, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{i=1}^{k} I(\gamma; a_i)\operatorname{ord}_f(a_i) - \sum_{j=1}^{\ell} I(\gamma; b_j)\operatorname{ord}_f(b_j)$$

Proof. Apply the residue theorem to $g = \frac{f'}{f}$. If $z_0 \in U$ is not a pole of f, then f and hence f' are holomorphic near z_0. If additionally z_0 is not a zero of f, g is holomorphic near z_0. So the set of singularities of g is precisely $\{a_1, \dots, a_k\} \cup \{b_1, \dots, b_\ell\}$. By the previous proposition, their residues are known, and the result follows. □

Remark. Let f, γ be as in the theorem, and let $\Gamma(t) = f(\gamma(t))$. Then $\Gamma(t)$ is a closed curve with image $\operatorname{Im}\Gamma \subset \mathbb{C} \setminus \{0\}$, since no zeroes or poles of f are in $\operatorname{Im}\gamma$. Moreover, if $[a,b]$ is the domain of γ, we have

$$I(\Gamma; 0) = \frac{1}{2\pi i}\int_\Gamma \frac{\mathrm{d}z}{z} = \frac{1}{2\pi i}\int_a^b \frac{\Gamma'(t)}{\Gamma(t)}\,\mathrm{d}t = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))}\,\mathrm{d}t = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z$$

Thus, $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}$ is the number of times the image curve $f \circ \gamma$ winds around zero as we move along γ.

Definition. Let Ω be a domain, and let γ be a closed curve in $\mathbb{C}$. We say that γ *bounds* Ω if $I(\gamma; w) = 1$ for all $w \in \Omega$, and $I(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus (\Omega \cup \operatorname{Im}\gamma)$.

Example. $\partial D(0,1)$ bounds $D(0,1)$, but does not bound $D(0,1) \setminus \{0\}$.

Remark. If γ bounds Ω, then

(i) Ω is bounded. Indeed, let $D(a,R)$ such that $\operatorname{Im}\gamma \subseteq D(a,R)$. Then $I(\gamma; w) = 0$ for $w \in \mathbb{C} \setminus D(a,R)$. Since $I(\gamma; w) = 1$ for all $w \in \Omega$, we must have $\Omega \subset D(a,R)$.

(ii) the topological boundary $\partial\Omega$ is contained within $\operatorname{Im}\gamma$, but it need not be the case that $\partial\Omega = \operatorname{Im}\gamma$.

There is a large class of closed curves that bound domains, namely, *simple closed curves*, which are curves $\gamma\colon [a,b] \to \mathbb{C}$ with $\gamma(a) = \gamma(b)$, and such that $\gamma(t_1) = \gamma(t_2)$ implies $t_1 = t_2$ or $t_1, t_2 \in \{a,b\}$. That a simple closed curve bounds a domain is a highly non-trivial fact guaranteed by the

Jordan curve theorem: if γ is a simple closed curve, then $\mathbb{C} \setminus \operatorname{Im}\gamma$ consists precisely of two connected components, one of which is bounded and the other unbounded, and moreover, γ (or $-\gamma$) bounds the bounded component, and $\operatorname{Im}\gamma$ is the boundary of each of the two components. Thus, if Ω_1 is the bounded component and Ω_2 is the unbounded component, then after possibly changing the orientation of γ, we have $I(\gamma; w) = 1$ for $w \in \Omega_1$, and $I(\gamma; w) = 0$ for $w \in \Omega_2$. This last assertion is simply that for any disc $D(a, R) \supset \operatorname{Im}\gamma$, we have $I(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus D(a, R)$.

For a domain bounded by a closed curve, the argument principle gives the following.

Corollary. Let γ be a closed curve bounding a domain Ω, and let f be meromorphic in an open set U with $\Omega \cup \operatorname{Im}\gamma \subseteq U$. Suppose that f has no zeroes or poles on $\operatorname{Im}\gamma$. Then f has finitely many zeroes and finitely many poles in Ω.
Let the number of zeroes in Ω be N, and the number of poles in Ω be P, both counted with multiplicity. Then in addition we have that

$$N - P = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, \mathrm{d}z = I(\Gamma; 0)$$

where $\Gamma = f \circ \gamma$.

Proof. Since f is meromorphic in U, its singularities form a discrete set $S \subset U$ consisting of poles or removable singularities. Since γ bounds Ω, we have that Ω is bounded and hence $\overline{\Omega}$ is compact. Also, $\overline{\Omega} \subseteq \Omega \cup \operatorname{Im}\gamma \subseteq U$. If $\overline{\Omega} \cap S$ is infinite, then by compactness of $\overline{\Omega}$, there exists a point $w \in \overline{\Omega}$ and distinct points $w_j \in \overline{\Omega} \cap S$ such that $w_j \to w$. If $w \notin S$, then f is defined and holomorphic near w which is impossible since $w_j \in S$ and $w_j \to w$. So $w \in S$, but this is impossible since S is a discrete set. So $\overline{\Omega} \cap S$ is finite, and in particular P is finite.

If f has infinitely many zeroes in Ω, then by compactness there exists $z \in \overline{\Omega} \subset U$ and distinct zeroes $z_j \in \Omega$ such that $z_j \to z$. Then either $z \in U \setminus S$, or (if $z \in S$) z is a removable singularity, since otherwise z would be a pole and hence $|f(\zeta)| \to \infty$ as $\zeta \to z$ which is impossible since $z_j \to z$ and $f(z_j) = 0$. In either case, by the principle of isolated zeroes, f must be identically zero in $D(z, \rho) \setminus \{z\}$ for some $\rho > 0$. Since f is holomorphic in $\Omega \setminus S$ which is connected (since $\Omega \cap S$ is finite and Ω) is connected, it follows from the unique continuation principle that $f \equiv 0$ in Ω. This is impossible since f has no zeroes in $\operatorname{Im}\gamma$, so N must be finite.

By the definition of γ bounding Ω, we have that $I(\gamma; w) = 1$ for all $w \in \Omega$, and $I(\gamma; w) = 0$ for all $w \in \mathbb{C} \setminus (\Omega \cup \operatorname{Im}\gamma)$. In particular, γ is homologous to zero in U. The final conclusion then follows from the fact that Γ is a closed curve in $\mathbb{C} \setminus \{0\}$ and $I(\gamma; 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, \mathrm{d}z$ as proven above. □

5.2 Local degree theorem

Definition. Let f be a holomorphic function on a disc $D(a, R)$ that is not constant. Then the *local degree of f at a*, denoted $\deg_f(a)$, is the order of the zero of $f(z) - f(a)$ at $z = a$. This is a finite positive integer.

Example. If $f(z) = (z-1)^4 + 1$ has $\deg_f(1) = 4$.

Theorem. Let $f : D(a, R) \to \mathbb{C}$ be holomorphic and non-constant, with $\deg_f(a) = d$. Then there exists $r_0 > 0$ such that for any $r \in (0, r_0]$, there exists $\varepsilon > 0$ such that for all w with $0 < |f(a) - w| < \varepsilon$, the equation $f(z) = w$ has precisely d distinct roots in $D(a, r) \setminus \{a\}$.

Proof. Let $g(z) = f(z) - f(a)$. Since g is non-constant, $g' \not\equiv 0$ in $D(a, R)$. Applying the principle of isolated zeroes to g and g', there exists $r_0 \in (0, R)$ such that $g(z) \neq 0$ and $g'(z) \neq 0$ for $z \in \overline{D(a, r_0)} \setminus \{a\}$.

We will show that the conclusion holds for this choice of r_0. Let $r \in (0, r_0]$, and for $t \in [0, 1]$, let $\gamma(t) = a + re^{2\pi i t}$ and $\Gamma(t) = g(\gamma(t))$. Note that $\operatorname{Im} \Gamma$ is compact and hence closed in $\mathbb{C}$, and $0 \notin \operatorname{Im} \Gamma$ since $g \neq 0$ on $\partial D(a, r)$. Hence there exists $\varepsilon > 0$ such that $D(0, \varepsilon) \subseteq \mathbb{C} \setminus \operatorname{Im} \Gamma$.

We now show that this ε satisfies the conditions in the theorem for this r. Let w such that $0 < |w - f(a)| < \varepsilon$. Then $w - f(a) \in D(0, \varepsilon) \subseteq \mathbb{C} \setminus \operatorname{Im} \Gamma$. Since $z \mapsto I(\Gamma; z)$ is locally constant, it is constant on $D(0, \varepsilon)$, so in particular $I(\Gamma; w - f(a)) = I(\Gamma; 0)$.

By direct calculation,

$$I(\Gamma; w - f(a)) = \frac{1}{2\pi i} \int_0^1 \frac{g'(\gamma(t))\gamma'(t)}{g(\gamma(t)) - (w - f(a))} \, \mathrm{d}t = \frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f'(z)}{f(z) - w} \, \mathrm{d}z$$

By the argument principle, $I(\Gamma; 0) = d$, since $I(\Gamma; 0)$ is the number of zeroes of g in $D(a, r)$ counted with multiplicity; the zero of g at $z = a$ has order d, and it is the only zero in $D(a, r)$. Hence,

$$\frac{1}{2\pi i} \int_{\partial D(a,r)} \frac{f'(z)}{f(z) - w} \, \mathrm{d}z = d$$

Again, the argument principle shows that the number of zeroes of $f(z) - w$ in $D(a, r)$ is d, counted with multiplicity. None of these zeroes is equal to a since $w \neq f(a)$. Since $f'(z) = g'(z) \neq 0$ in $D(a, r) \setminus \{a\}$, it follows from the Taylor series that these zeroes are simple. Thus $f(z) - w$ has d distinct zeroes in $D(a, r) \setminus \{a\}$. □

5.3 Open mapping theorem

Corollary. A non-constant holomorphic function maps open sets to open sets. That is, non-constant holomorphic functions are open maps.

Proof. Let $f : U \to \mathbb{C}$ be holomorphic and non-constant, and let $V \subseteq U$ be an open set. Let $b \in f(V)$. Then $b = f(a)$ for some $a \in V$. Since V is open, there exists $r > 0$ such that $D(a, r) \subseteq V$. By the local degree theorem, if r is sufficiently small, there exists $\varepsilon > 0$ such that $w \in D(f(a), \varepsilon) \setminus \{f(a)\} \implies w = f(z)$ for some $z \in D(a, r) \setminus \{a\}$, hence $D(f(a), \varepsilon) \setminus \{f(a)\} \subseteq f(D(a, r) \setminus \{a\})$. Hence $D(b, \varepsilon) = D(f(a), \varepsilon) \subseteq f(D(a, r)) \subseteq f(V)$. Thus, for all $b \in f(V)$, there exists a disc $D(b, \varepsilon) \subseteq f(V)$, so $f(V)$ is open. □

5.4 Rouché's theorem

Theorem. Let γ be a closed curve bounding a domain Ω, and let f, g be holomorphic functions on an open set U containing $\Omega \cup \operatorname{Im}\gamma$. If $|f(z) - g(z)| < |g(z)|$ for all $z \in \operatorname{Im}\gamma$, then f and g have the same number of zeroes in Ω, counted with multiplicity.

Proof. The strict inequality $|f - g| < |g|$ on $\operatorname{Im}\gamma$ implies that f, g are never zero on $\operatorname{Im}\gamma$ and hence never zero on some open set V containing $\operatorname{Im}\gamma$. So $h = \frac{f}{g}$ is holomorphic and nonzero in V. In particular, g is not identically zero in Ω, and hence the zeroes of g in $\Omega \cup V$ are isolated. Hence h is meromorphic in $\Omega \cup V$, and h has no zeroes or poles on $\operatorname{Im}\gamma$. Also, f, g have finitely many zeroes in Ω.

Now, $|h(z) - 1| < 1$ for all $z \in \operatorname{Im}\gamma$. Hence, the curve $\Gamma = h \circ \gamma$ has image contained within $D(1, 1)$. Since zero is outside this disc, $I(\Gamma; 0) = 0$, and so by the argument principle,

$$\sum_{w \in \mathcal{P}} \operatorname{ord}_h(w) = \sum_{w \in \mathcal{Z}} \operatorname{ord}_h(w)$$

where $\mathcal{P}$ and $\mathcal{Z}$ denote the sets of distinct poles and zeroes of h respectively, and the sums are finite. Now, $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$ and $\mathcal{Z} = \mathcal{Z}_1 \cup \mathcal{Z}_2$, where

$$\begin{aligned}
\mathcal{P}_1 &= \{w \in \Omega : \, g(w) = 0; f(w) \neq 0\}; \\
\mathcal{P}_2 &= \left\{w \in \Omega : \, g(w) = f(w) = 0; \operatorname{ord}_g(w) > \operatorname{ord}_f(w)\right\}; \\
\mathcal{Z}_1 &= \{w \in \Omega : \, f(w) = 0; g(w) \neq 0\}; \\
\mathcal{Z}_2 &= \left\{w \in \Omega : \, f(w) = g(w) = 0; \operatorname{ord}_f(w) > \operatorname{ord}_g(w)\right\}
\end{aligned}$$

Hence,

$$\sum_{w \in \mathcal{P}_1} \operatorname{ord}_g(w) + \sum_{w \in \mathcal{P}_2} \left(\operatorname{ord}_g(w) - \operatorname{ord}_f(w)\right) = \sum_{w \in \mathcal{Z}_1} \operatorname{ord}_f(w) + \sum_{w \in \mathcal{Z}_2} \left(\operatorname{ord}_f(w) - \operatorname{ord}_g(w)\right)$$

Equivalently,

$$\sum_{w \in \mathcal{P}_1} \operatorname{ord}_g(w) + \sum_{w \in \mathcal{P}_2} \operatorname{ord}_g(w) + \sum_{w \in \mathcal{Z}_2} \operatorname{ord}_g(w) = \sum_{w \in \mathcal{Z}_1} \operatorname{ord}_f(w) + \sum_{w \in \mathcal{Z}_2} \operatorname{ord}_f(w) + \sum_{w \in \mathcal{P}_2} \operatorname{ord}_f(w)$$

Adding $\sum_{w \in \mathcal{R}} \operatorname{ord}_g(w)$ to the left hand side and the equal number $\sum_{w \in \mathcal{R}} \operatorname{ord}_f(w)$ to the right hand side, where

$$\mathcal{R} = \left\{w \in \Omega : \, f(w) = g(w) = 0; \operatorname{ord}_f(w) = \operatorname{ord}_g(w)\right\}$$

we have

$$\sum_{w \in \Omega : \, g(w) = 0} \operatorname{ord}_g(w) = \sum_{w \in \Omega : \, f(w) = 0} \operatorname{ord}_f(w)$$

as required. □

Example. $z^4 + 6z + 3$ has three roots counted with multiplicity in $\{1 < |z| < 2\}$. Let $f(z) = z^4 + 6z + 3$.

On $|z| = 2$ we have $|z^4| = 16$ and $|6z + 3| \leq 6|z| + 3 = 15$, so $|z|^4 > |6z + 3|$. By Rouché's theorem, f has the same number of roots inside $\{|z| < 2\}$ as z^4, counting with multiplicity. Thus, all roots of $z^4 + 6z + 3$ lie inside $\{|z| < 2\}$; this is all of the roots since f is a polynomial with degree 4.

On $|z| = 1$, we have $|6z| = 6$ and $|z^4 + 3| \leq |z|^4 + 3 \leq 4$. Again by Rouché's theorem, f has one root inside $\{|z| < 1\}$, as $6z$ has one root in this region. From the strict inequalities, no roots lie on $\{|z| = 2\}$ or $\{|z| = 1\}$. Hence three roots of f lie in $|z \in \mathbb{C} : \, 1 < |z| < 2|$.